# BOUNDARY VALUE PROBLEMS

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## *Boundary Value Problems*

We determine $\vec{E}$ using

1. Coulomb's law
2. Gauss's law

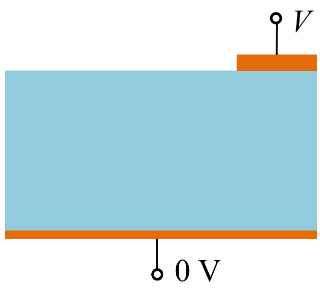
$\Big\}$ $\rightarrow$ when charge distribution is known.

3. $\vec{E} = -\nabla V$ $\rightarrow$ when potential distribution is known.

**Consider** $\rightarrow$ Electrostatic conditions only at some boundaries.

$V = ?$

$E = ?$

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## *Poisson's & Laplace's Equations*

From Gauss's law

$$\nabla \cdot \vec{D} = \nabla \cdot \varepsilon \vec{E} = \rho_v$$

And $\quad \vec{E} = -\nabla V$

Therefore $\nabla \cdot \left(-\varepsilon \nabla V\right) = \rho_v$ for inhomogeneous medium

For homogeneous medium $\boxed{\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}}$ $\rightarrow$ Poisson's Equation

Laplace's Equation $\boxed{\nabla^2 V = 0}$ $\rightarrow$ For charge free medium

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## *Laplace's Equation*

In Cartesian coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In Cylindrical coordinates

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In Spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin 2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

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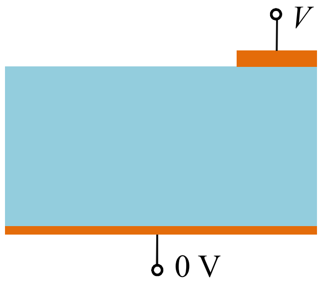
## *Solving Laplace's Equation*

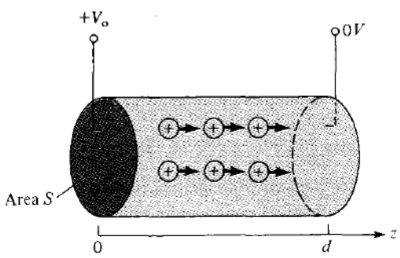
1. Solve Laplace's equation using
   - Direct integration, where $V$ is a function of one variable
   - Separation of variables if $V$ is a function of more than one variable
2. Apply boundary conditions
3. Find $\vec{E}$ using $\vec{E} = -\nabla V$
4. Find $Q$, $C$, etc.






## *Example – 1*

The region between the electrodes contains a uniform charge $\rho_0$, which is generated at the left electrode and collected at the right electrode. Calculate the potential in the region.



# Example – 1

**Solve Poisson's equation**

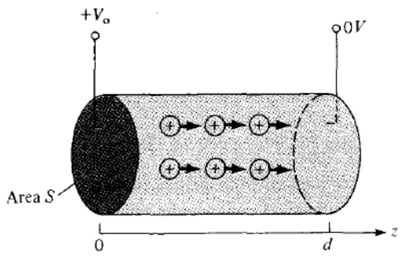
$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

*V* depends only on *z*

$$\frac{d^2V}{dz^2} = -\frac{\rho_0}{\varepsilon}$$

$$\frac{dV}{dz} = -\frac{\rho_0 z}{\varepsilon} + A$$

$$V = -\frac{\rho_0 z^2}{2\varepsilon} + Az + B$$

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# Example – 1

**Apply boundary conditions**

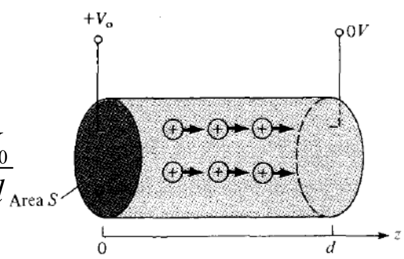
When $z = 0$, $V = V_0$

$$V_0 = -0 + 0 + B \rightarrow B = V_0$$

When $z = d$, $V = 0$

$$0 = -\frac{\rho_0 d^2}{2\varepsilon} + Ad + V_0 \rightarrow A = \frac{\rho_0 d}{2\varepsilon} - \frac{V_0}{d}$$

$$V = -\frac{\rho_0 z^2}{2\varepsilon} + \left(\frac{\rho_0 d}{2\varepsilon} - \frac{V_0}{d}\right) z + V_0$$

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