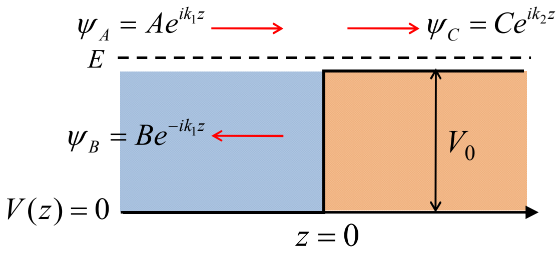
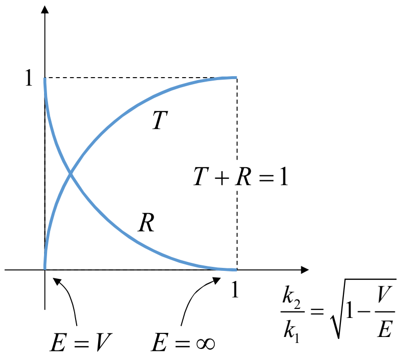
TUNNELING TRANSPORT

1

Which one is you?

Reflection and Transmission





- **Case I: $E > V_0$**

$$\text{Reflection} = R = \left|\frac{B}{A}\right|^2 = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2$$

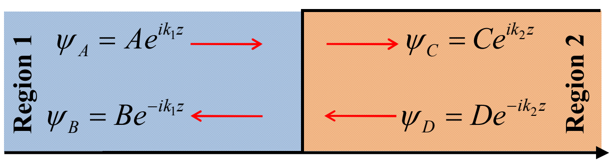
- **Case II: $E < V_0$**

$$\text{Reflection} = R = \left|\frac{B}{A}\right|^2 = 1$$

S-Matrices

- What comes out as a function of what goes in.

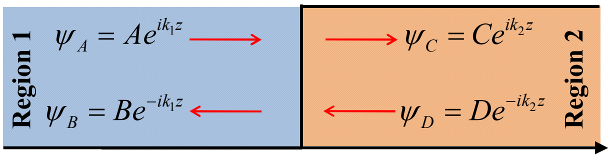
$$\begin{pmatrix} C \\ B \end{pmatrix} = S \begin{pmatrix} A \\ D \end{pmatrix}$$



T-Matrices

- The waves on the right as a function of those on the left.

$$\begin{pmatrix} C \\ D \end{pmatrix} = T^{(21)} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} T_{11}^{(21)} & T_{12}^{(21)} \\ T_{21}^{(21)} & T_{22}^{(21)} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$



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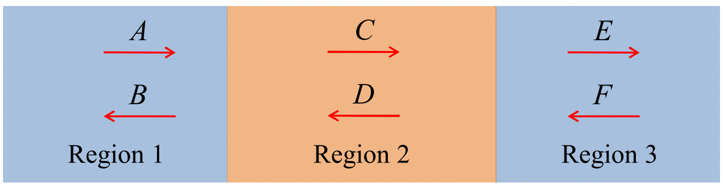
More Than Two Regions



$$\begin{pmatrix} C \\ D \end{pmatrix} = T^{(21)} \begin{pmatrix} A \\ B \end{pmatrix}, \qquad \begin{pmatrix} E \\ F \end{pmatrix} = T^{(32)} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$\begin{pmatrix} E \\ F \end{pmatrix} = T^{(32)} T^{(21)} \begin{pmatrix} A \\ B \end{pmatrix} = T^{(31)} \begin{pmatrix} A \\ B \end{pmatrix}$$

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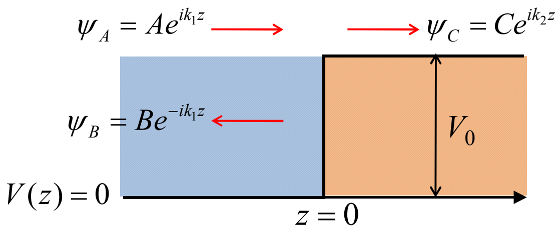
Reflection and Transmission

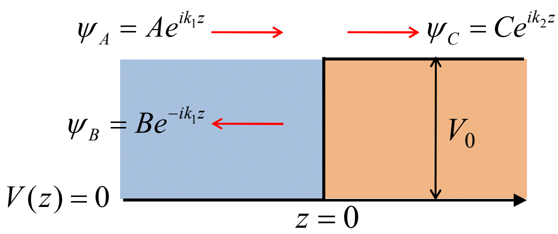


$$\begin{pmatrix} C \\ D \end{pmatrix} = T \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = T \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$r = B/A \qquad t = C/A$$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix}$$

Reflection and Transmission



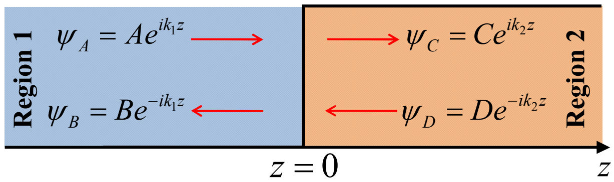
$$\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$t = T_{11} + T_{12} r$$

$$0 = T_{21} + T_{22} r \Rightarrow r = -T_{21}/T_{22}$$

$$t = \left(T_{11} T_{22} - T_{12} T_{21}\right)/T_{22}$$

T-Matrix for Step Potential



$$\psi_1 = Ae^{ik_1 z} + Be^{-ik_1 z} \qquad \psi_2 = Ce^{ik_2 z} + De^{-ik_2 z}$$

- ψ is continuous: $\psi_1(0) = \psi_2(0)$ ⇒ $A + B = C$
- $\frac{\partial \psi}{\partial z}$ is continuous: $\frac{\partial}{\partial z}\psi_1(0) = \frac{\partial}{\partial z}\psi_2(0)$ ⇒ $k_1(A - B) = k_2(C - D)$

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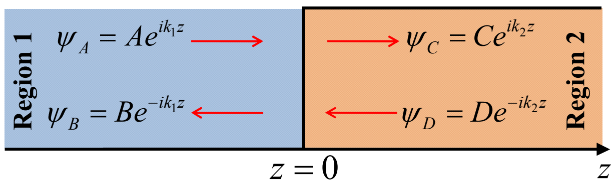
T-Matrix for Step Potential



$$C = \frac{1}{2}\left(1 + \frac{k_1}{k_2}\right)A + \frac{1}{2}\left(1 - \frac{k_1}{k_2}\right)B$$

$$D = \frac{1}{2}\left(1 - \frac{k_1}{k_2}\right)A + \frac{1}{2}\left(1 + \frac{k_1}{k_2}\right)B$$

$$T^{(21)} = \frac{1}{2k_2}\begin{pmatrix} k_2 + k_1 & k_2 - k_1 \\ k_2 - k_1 & k_2 + k_1 \end{pmatrix} \equiv T(k_2, k_1)$$

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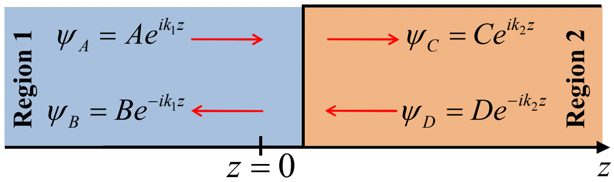
Barrier Away from Origin

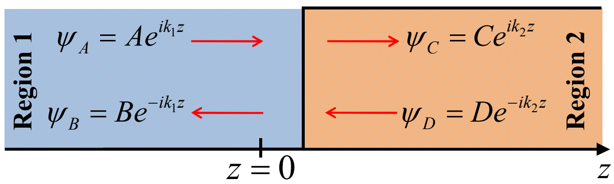


- Let $z' = z - d$

$$\psi_1 = Ae^{ik_1 z'} e^{ik_1 d} + Be^{-ik_1 z'} e^{-ik_1 d}$$

$$\psi_2 = Ce^{ik_2 z'} e^{ik_2 d} + De^{-ik_2 z'} e^{-ik_2 d}$$

Barrier Away from Origin

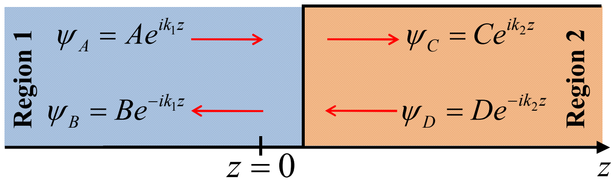


- Apply boundary conditions and solve C and D in terms of A and B

$$C = \frac{1}{2} e^{-ik_2 d} \left(1 + k_1 / k_2\right) Ae^{ik_1 d} + \frac{1}{2} e^{-ik_2 d} \left(1 - k_1 / k_2\right) Be^{-ik_1 d}$$

$$D = \frac{1}{2} e^{ik_2 d} \left(1 - k_1 / k_2\right) Ae^{ik_1 d} + \frac{1}{2} e^{-ik_2 d} \left(1 + k_1 / k_2\right) Be^{-ik_1 d}$$

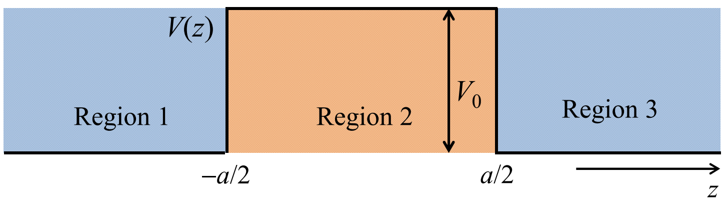
T-Matrix



$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} e^{-ik_2 d} & 0 \\ 0 & e^{ik_2 d} \end{pmatrix} \frac{1}{2k_2} \begin{pmatrix} k_2 + k_1 & k_2 - k_1 \\ k_2 - k_1 & k_2 + k_1 \end{pmatrix} \begin{pmatrix} e^{ik_1 d} & 0 \\ 0 & e^{-ik_1 d} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$T(d) = \begin{pmatrix} e^{-ik_2 d} & 0 \\ 0 & e^{ik_2 d} \end{pmatrix} T(0) \begin{pmatrix} e^{ik_1 d} & 0 \\ 0 & e^{-ik_1 d} \end{pmatrix}$$

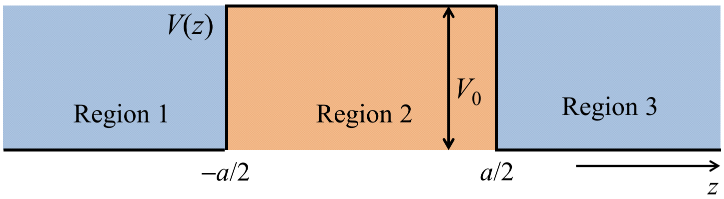
Square Barrier



$$T^{(31)} = \begin{pmatrix} e^{-ik_1 a/2} & 0 \\ 0 & e^{ik_1 a/2} \end{pmatrix} T\left(k_1, k_2\right) \begin{pmatrix} e^{ik_2 a/2} & 0 \\ 0 & e^{-ik_2 a/2} \end{pmatrix}$$

$$\times \begin{pmatrix} e^{ik_2 a/2} & 0 \\ 0 & e^{-ik_2 a/2} \end{pmatrix} T\left(k_2, k_1\right) \begin{pmatrix} e^{-ik_1 a/2} & 0 \\ 0 & e^{ik_1 a/2} \end{pmatrix}$$

Square Barrier



$$t = \frac{T_{11}T_{22} - T_{12}T_{21}}{T_{22}} = \frac{2k_1k_2 e^{-ik_1 a}}{2k_1k_2 \cos k_2 a - i\left(k_1^2 + k_2^2\right)\sin k_2 a}$$

$$E > V_0 \rightarrow k_2 = \left[2m\left(E - V_0\right)/\hbar^2\right]^{1/2}$$

$$T = \frac{4k_1^2k_2^2}{4k_1^2k_2^2 + \left(k_1^2 - k_2^2\right)^2 \sin^2 k_2 a} = \left[1 + \frac{V_0^2}{4E\left(E - V_0\right)}\sin^2 k_2 a\right]^{-1}$$

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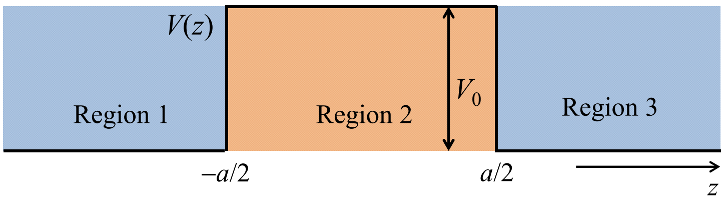
Square Barrier



$$t = \frac{T_{11}T_{22} - T_{12}T_{21}}{T_{22}} = \frac{2k_1k_2 e^{-ik_1 a}}{2k_1k_2 \cos k_2 a - i\left(k_1^2 + k_2^2\right)\sin k_2 a}$$

$$E < V_0 \rightarrow k_2 = i\kappa_2$$

$$T = \frac{4k_1^2\kappa_2^2}{4k_1^2\kappa_2^2 + \left(k_1^2 + \kappa_2^2\right)^2 \sin^2 \kappa_2 a} = \left[1 + \frac{V_0^2}{4E\left(V_0 - E\right)}\sinh^2 \kappa_2 a\right]^{-1}$$

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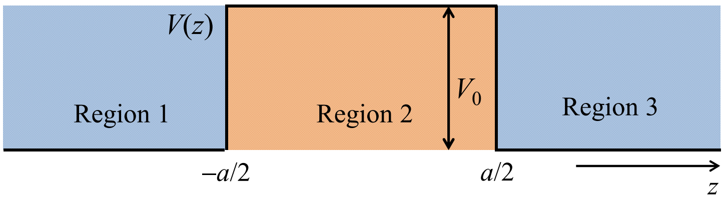
Square Barrier

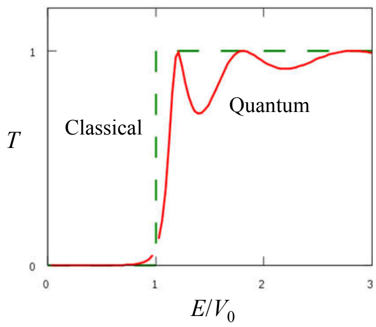


$$E > V_0$$

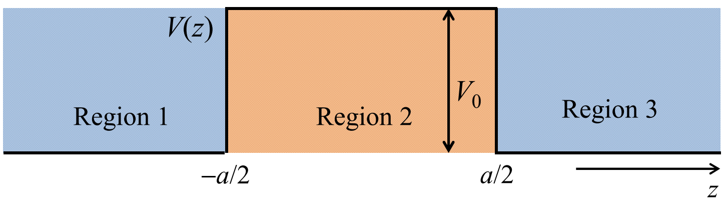
$$T = \left[1 + \frac{V_0^2}{4E(E - V_0)}\sin^2 k_2 a\right]^{-1}$$

$$E < V_0 \rightarrow k_2 = i\kappa_2$$

$$T = \left[1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2 \kappa_2 a\right]^{-1}$$



Square Barrier

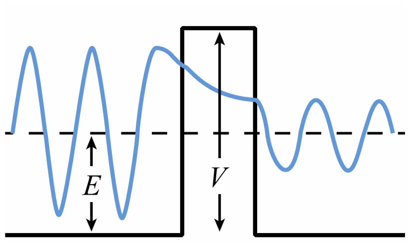


$$E > V_0$$

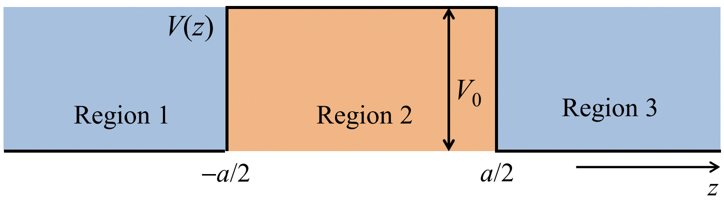
$$T = \left[1 + \frac{V_0^2}{4E(E - V_0)}\sin^2 k_2 a\right]^{-1}$$

$$E < V_0 \rightarrow k_2 = i\kappa_2$$

$$T = \left[1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2 \kappa_2 a\right]^{-1}$$



Square Barrier



$$E < V_0 \rightarrow k_2 = i\kappa_2$$

$$T = \left[1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2 \kappa_2 a\right]^{-1}$$

$$\sinh \kappa_2 a = \frac{1}{2}\left[e^{\kappa_2 a} - e^{-\kappa_2 a}\right]$$

$$\sinh^2 \kappa_2 a = \frac{1}{4}\left[e^{\kappa_2 a} - e^{-\kappa_2 a}\right]^2 \approx \frac{1}{4}e^{2\kappa_2 a}$$

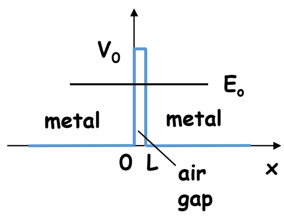
$$T = \frac{1}{1 + \dfrac{V_0^2}{16E(V_0 - E)}e^{2\kappa_2 a}}$$

$$T \approx \frac{16E(V_0 - E)}{V_0^2}e^{-2\kappa_2 a}$$

Example: Barrier Tunneling

- Let's consider a tunneling problem:

An electron with a total energy of $E_0 = 6$ eV approaches a potential barrier with a height of $V_0 = 12$ eV. If the width of the barrier is $L = 0.18$ nm, what is the probability that the electron will tunnel through the barrier?



$$T \approx \frac{16E_0(V_0 - E_0)}{V_0^2}e^{-2\kappa L}$$

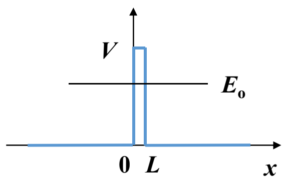
$$\kappa = \sqrt{\frac{2m_e}{\hbar^2}(V_0 - E_0)} = 2\pi\sqrt{\frac{2m_e}{h^2}(V_0 - E_0)} = 2\pi\sqrt{\frac{6\,\text{eV}}{1.505\,\text{eV-nm}^2}} \approx 12.6\,\text{nm}^{-1}$$

$$T = 4e^{-2(12.6\,\text{nm}^{-1})(0.18\,\text{nm})} = 4(0.011) = \boxed{4.4\%}$$

Question: What will *T* be if we double the width of the barrier?

Multiple Choice Questions

Consider a particle tunneling through a barrier:



1. Which of the following will increase the likelihood of tunneling?
   a. decrease the height of the barrier
   b. decrease the width of the barrier
   c. decrease the mass of the particle

2. What is the energy of the particles that have successfully "escaped"?
   a. < initial energy
   b. = initial energy
   c. > initial energy

Although the *amplitude* of the wave is smaller after the barrier, no energy is lost in the tunneling process

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Class Test – 4

Day: 05 February 2019

Syllabus: Lectures 23 and 24

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