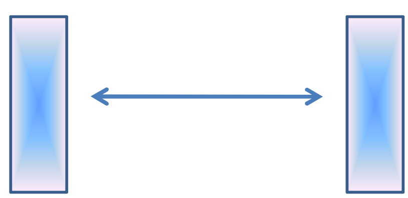
RESONATORS

Resonators

- A key ingredient to our laser oscillators is a feedback mechanism now that we have gain/amplification.
- The simplest feedback can be provided by mirrors.
- In E&M, we can terminate a transmission line by shorting it out and achieve line resonances. The equivalent to shorting out the ends of the transmission line would be to have highly reflecting mirrors to contain the light.



Mirrors

- We will analyze for simplicity plane parallel mirrors. This is called a Fabry-Perot resonator.
- Fabry-Perot resonators are not the most stable ones since if the mirrors are slightly tilted the light walks-off.
- Curved mirrors are often used in a more stable configuration.
- Semiconductor lasers typically have cleaved facets and thus are like the classic plane-parallel resonator.

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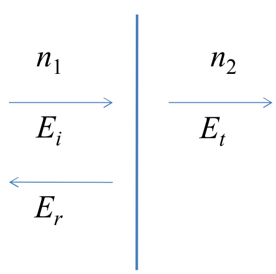
Reviewing E&M



$$E_t = \frac{2n_1}{n_1 + n_2} E_i$$

$$E_r = \frac{n_1 - n_2}{n_1 + n_2} E_i$$

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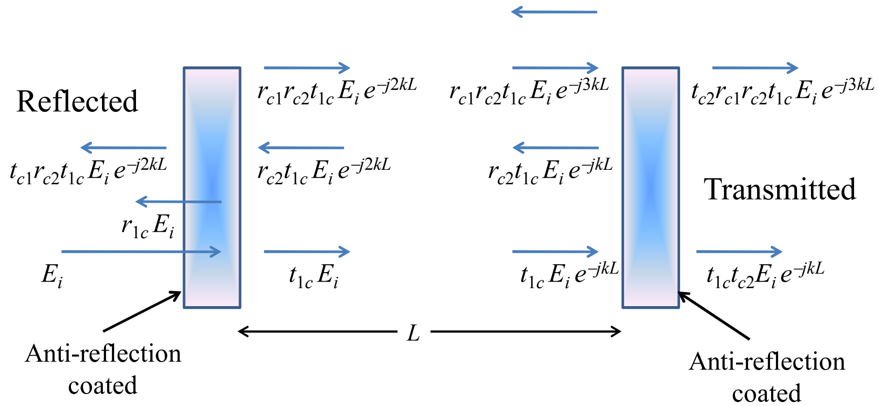
Two Mirrors



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Transmitted

$$\frac{E_{\text{out}}}{E_i} = t_{c2}t_{1c}e^{-jkL} + t_{c2}r_{c1}r_{c2}t_{1c}e^{-3jkL} + t_{c2}r_{c1}r_{c2}r_{c1}r_{c2}t_{1c}e^{-5jkL} + \cdots$$

$$= t_{c2}t_{1c}e^{-jkL}\sum_{n=0}^{\infty}\left(r_{c1}r_{c2}e^{-2jkL}\right)^n$$

$$= \frac{t_{c2}t_{1c}e^{-jkL}}{1 - r_{c1}r_{c2}e^{-2jkL}}$$

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Reflected

$$\frac{E_{\text{reflected}}}{E_i} = r_{1c} + t_{c1}r_{c2}t_{1c}e^{-2jkL} + t_{c1}r_{c2}r_{c1}r_{c2}t_{1c}e^{-4jkL} + \cdots$$

$$= r_{1c} + \frac{t_{c1}r_{c2}t_{1c}e^{-2jkL}}{1 - r_{c1}r_{c2}e^{-2jkL}}$$

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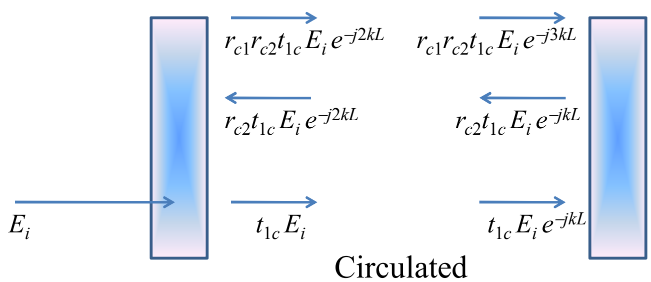
Circulated



$$\frac{E_{\text{circ}}}{E_i} = t_{1c} + r_{c1}r_{c2}t_{1c}e^{-2jkL} + r_{c1}r_{c2}r_{c1}r_{c2}t_{1c}e^{-4jkL} + \cdots$$

$$= \frac{t_{1c}}{1 - r_{c1}r_{c2}e^{-2jkL}}$$

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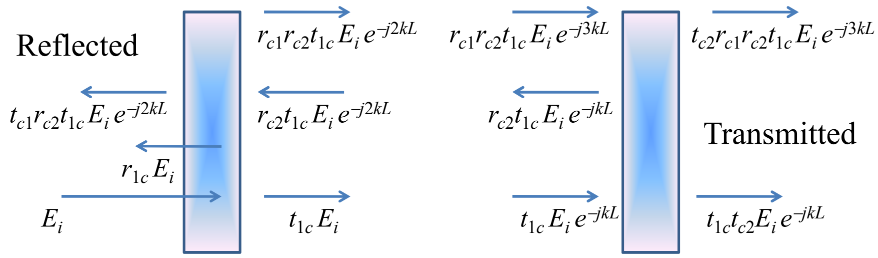
Coefficients



$$\frac{E_{t1c}}{E_i} = t_{1c} = \frac{2n_1}{n_1+n_2}$$

$$\frac{E_{tc2}}{E_i} = t_{c2} = \frac{2n_2}{n_1+n_2}$$

$$\frac{E_{r1c}}{E_i} = r_{1c} = \frac{n_1-n_2}{n_1+n_2} = -r_{c1} = -r_{c2}$$

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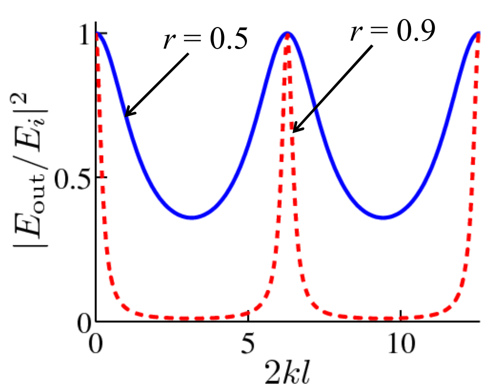
Transmitted Intensity

$$t_{c2}t_{1c} = 1-r^2, \quad r = |r_{c1}| = |r_{c2}|$$

$$\frac{E_{\text{out}}}{E_i} = \frac{t_{c2}t_{1c}e^{-jkl}}{1-r_{c1}r_{c2}e^{-2jkl}} = \frac{(1-r^2)e^{-jkl}}{1-r^2e^{-2jkl}}$$

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{1-r^2}{1-r^2e^{-2jkl}}\frac{1-r^2}{1-r^2e^{2jkl}} = \frac{(1-r^2)^2}{1+r^4-2r^2\cos 2kL}$$



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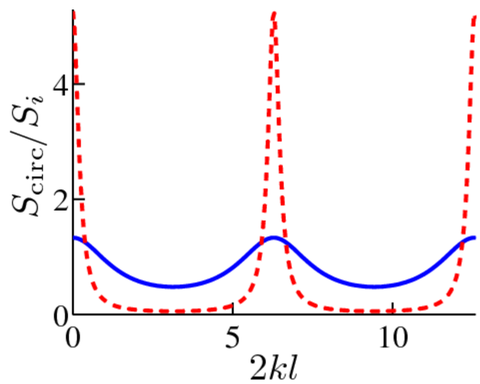
Circulated Intensity

$$\left|\frac{E_{\text{circ}}}{E_i}\right|^2 = \frac{S_{\text{circ}}}{S_i}$$

$$= \frac{1-r^2}{1+r^4-2r^2\cos 2kL}$$

Note that this looks like the transmitted intensity but the numerator is different.



Peak Value

$$\frac{S_{\text{circ}}}{S_i} = \frac{1-r^2}{1+r^4-2r^2\cos 2kL}$$

$$\left.\frac{S_{\text{circ}}}{S_i}\right|_{\max} = \frac{1-r^2}{\left(1-r^2\right)^2} = \frac{1}{1-r^2} > 1$$

Thus the cavity builds up a circulating field. Note that we have calculated only the forward direction and not the backward direction. The field build-up is due to the fact that the cavity stores energy in the resonator. Here we are calculating steady-state build up. If we suddenly turn off the input the field will decay away.

Internal Loss

Suppose there is internal loss in the cavity. This could be absorption due to the background dielectric material, scattering losses, etc. If we have a field loss per unit length of α, then

$$\frac{E_{\text{out}}}{E_i} = t_{c2}t_{1c}e^{-jkL}e^{-\alpha L} + r_{c1}r_{c2}t_{c2}t_{1c}e^{-3jkL}e^{-3\alpha L} + r_{c1}r_{c2}r_{c1}r_{c2}t_{c2}t_{1c}e^{-5jkL}e^{-5\alpha L} + \cdots$$

$$= t_{c2}t_{1c}e^{-jkL}e^{-\alpha L}\left[\sum_{n=0}^{\infty}\left(r_{c1}r_{c2}e^{-2jkL}e^{-2\alpha L}\right)^n\right] = \frac{t_{c2}t_{1c}e^{-jkL}e^{-\alpha L}}{1-r_{c1}r_{c2}e^{-2jkL}e^{-2\alpha L}}$$

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{\left(1-r^2\right)^2 e^{-2\alpha L}}{1+r^4e^{-4\alpha L} - 2r^2e^{-2\alpha L}\cos 2kL}$$

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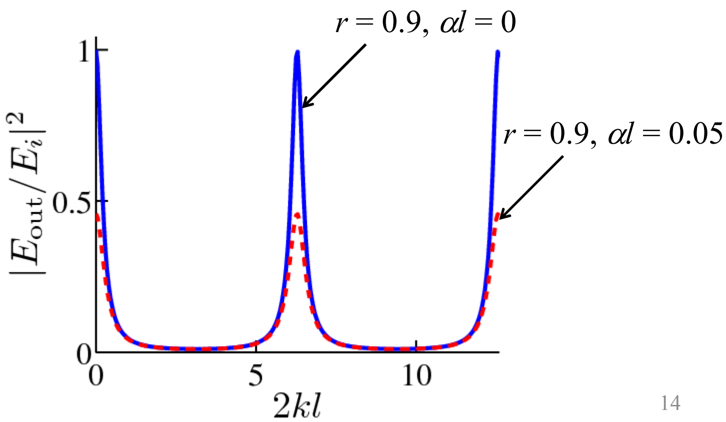
Transmitted Intensity

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{\left(1-r^2\right)^2 e^{-2\alpha L}}{1+r^4e^{-4\alpha L} - 2r^2e^{-2\alpha L}\cos 2kL}$$



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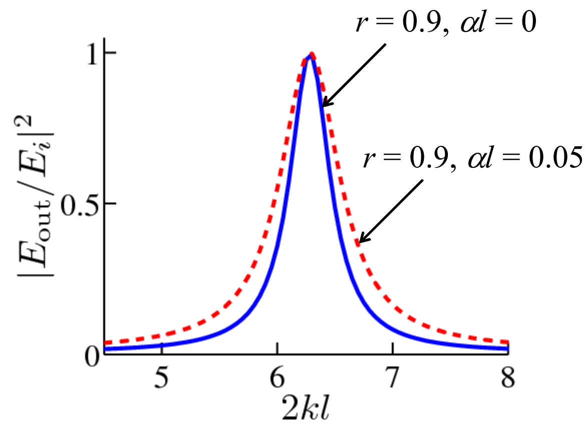
Bandwidth

Normalized:



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RESONANT MODES

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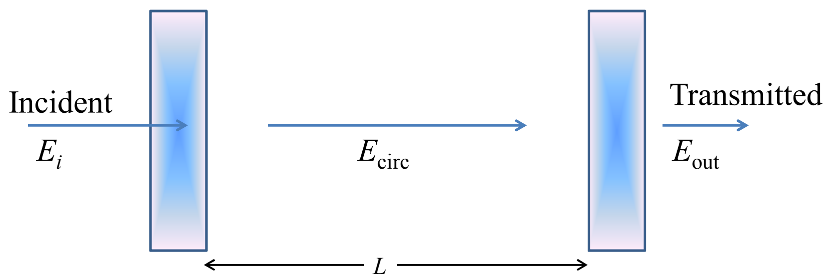
Two Mirrors



$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{\left(1-r^2\right)^2}{1+r^4-2r^2\cos 2kL}$$

$$\left|\frac{E_{\text{circ}}}{E_i}\right|^2 = \frac{1-r^2}{1+r^4-2r^2\cos 2kL}$$

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Resonances

- Resonant frequencies spaced evenly if medium has no dispersion.

$$2kL = 2m\pi \Rightarrow \left(\frac{n\omega}{c}\right)L = m\pi$$

$$\left(\frac{2\pi n\nu_m}{c}\right)L = m\pi$$

$$\nu_m = m\left[\frac{c}{2nL}\right]$$

$$\nu_{m+1} - \nu_m = \frac{c}{2nL}$$

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Index of Refraction

- Note that the resonances are changed by the index of refraction *n* and if *n* is a function of frequency the resonances may not be equally spaced!
- Thus the spacing of adjacent modes is $\Delta f = c/(2nL)$ if there is no frequency dependence.
- If there is frequency dependence then we note that resonance is defined as $2kL = m2\pi$. Thus we can write where we take a perturbation approach for *n* if justified.

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Mode Spacing

$$2\frac{\omega_a}{c}n(\omega_a)L = 2m\pi, \quad 2\frac{\omega_{a+1}}{c}n(\omega_{a+1})L = 2(m+1)\pi$$

$$\frac{2L}{c}\left[n(\omega_{a+1})\omega_{a+1} - n(\omega_a)\omega_a\right] = 2\pi$$

$$n(\omega_{a+1}) - n(\omega_a) \approx \left.\frac{dn}{d\omega}\right|_{\omega_a}\left(\omega_{a+1} - \omega_a\right) \Rightarrow$$

$$\frac{2Ln(\omega_a)}{c}\left(\omega_{a+1} - \omega_a\right) + \frac{2L\omega_{a+1}}{c}\left.\frac{dn}{d\omega}\right|_{\omega_a}\left(\omega_{a+1} - \omega_a\right) = 2\pi$$

$$\left(\omega_{a+1} - \omega_a\right)\left[\frac{2Ln(\omega_a)}{c} + \frac{2L\omega_{a+1}}{c}\left.\frac{dn}{d\omega}\right|_{\omega_a}\right] = 2\pi$$

$$\omega_{a+1} - \omega_a = 2\pi\frac{c}{2Ln(\omega_a)}\left[\frac{1}{1 + \frac{\omega_{a+1}}{n(\omega_a)}\frac{dn}{d\omega}}\right]$$

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Tuning of Cavity

We can tune the cavity resonance frequency by a variety of techniques.

- Change cavity length.
- Change index of refraction.
- Change index of refraction and/or length by changing temperature.

CAVITY GAINS AND LOSSSES

Delta Notation

- Mirror reflectivities

$$R = r^2 = e^{-\delta_r}$$
$$R \approx 1 - \delta_r \rightarrow \delta_r = T$$

- Alternatively

$$\delta_r = \ln\left(\frac{1}{R}\right) = 2\ln\left(\frac{1}{r}\right)$$

- Cavity losses and gain

$$\delta_0 = 2\alpha_0 L, \qquad \delta_m = 2\alpha_m L$$

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Transmitted Intensity

- We can express the transmitted intensity using the delta notations.

$$\frac{E_{\text{out}}}{E_i} = \frac{t_{c2}t_{1c}e^{-jkL}e^{-\alpha L}}{1 - r_{c1}r_{c2}e^{-2jkL}e^{-2\alpha L}}$$

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{\left(1-r^2\right)^2 e^{-2\alpha L}}{1 + r^4 e^{-4\alpha L} - 2r^2 e^{-2\alpha L}\cos 2kL}$$

$$\delta_0, \delta_r \ll 1 \;\&\; 2kL = 2m\pi$$

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{\left(\delta_r\right)^2 e^{-\delta_0}}{1 + e^{-2\delta_r}e^{-2\delta_0} - 2e^{-\delta_r}e^{-\delta_0}} = \frac{\left(\delta_r\right)^2 e^{-\delta_0}}{\left(1 - e^{-\delta_r}e^{-\delta_0}\right)^2} \approx \frac{\left(\delta_r\right)^2}{\left(\delta_r + \delta_0\right)^2}$$

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Circulation and Reflection

$$\frac{S_{\text{circ}}}{S_i} = \frac{1-r^2}{1+r^4e^{-4\alpha L}-2r^2e^{-2\alpha L}\cos 2kL} \approx \frac{\delta_r}{1+e^{-2\delta_r}e^{-2\delta_0}-2e^{-\delta_r}e^{-\delta_0}}$$

$$= \frac{\delta_r}{\left(1-e^{-\delta_r}e^{-\delta_0}\right)^2} \approx \frac{\delta_r}{\left(\delta_r+\delta_0\right)^2}$$

$$\frac{E_{\text{reflected}}}{E_i} = -r + \frac{r\left(1-r^2\right)e^{-2\alpha L}e^{-2jkL}}{1-r^2e^{-2\alpha L}e^{-2jkL}} \to -r + \frac{r\left(1-r^2\right)e^{-2\alpha L}}{1-r^2e^{-2\alpha L}} \approx -r + \frac{r\delta_r\left(1-\delta_0\right)}{1-e^{-(\delta_r+\delta_0)}}$$

$$= r\left[-1+\frac{\delta_r\left(1-\delta_0\right)}{\delta_r+\delta_0}\right] \approx r\left[\frac{-\delta_0}{\delta_r+\delta_0}\right] = e^{-\frac{\delta_r}{2}}\left[\frac{-\delta_0}{\delta_r+\delta_0}\right] \approx \frac{-\delta_0}{\delta_r+\delta_0}$$

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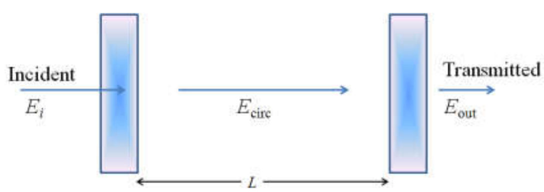
Round-Trip Gain, g_{rt}



$$\frac{E_{\text{out}}}{E_i} = \frac{t_{c2}t_{1c}e^{-jkL}e^{-\alpha L}}{1-r_{c1}r_{c2}e^{-2jkL}e^{-2\alpha L}}$$

$$\tilde{g}_{\text{rt}}(\omega) = r_{c1}r_{c2}e^{-2jkL}e^{-2\alpha L} = g_{\text{rt}}e^{-2jkL}$$

$$\frac{E_{\text{out}}}{E_i} = \frac{t_{c2}t_{1c}}{\sqrt{r_{c1}r_{c2}}}\frac{\sqrt{\tilde{g}_{\text{rt}}(\omega)}}{1-\tilde{g}_{\text{rt}}(\omega)}$$

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{\left(1-r^2\right)^2}{r^2}\frac{g_{\text{rt}}}{1+g_{\text{rt}}^2-2g_{\text{rt}}\cos 2kL}$$

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Bandwidth

If we consider that the losses are small, including mirror losses, then $g \approx 1$ and we can write

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2 = \frac{\left(1-r^2\right)^2}{r^2}\frac{g_{\text{rt}}}{1+g_{\text{rt}}^2-2g_{\text{rt}}\cos 2\frac{\omega}{c}L}$$

$$\left|\frac{E_{\text{out}}}{E_i}\right|^2_{\max} = \frac{\left(1-r^2\right)^2}{r^2}\frac{g_{\text{rt}}}{\left(1-g_{\text{rt}}\right)^2}$$

$$\frac{1}{2}\left|\frac{E_{\text{out}}}{E_i}\right|^2_{\max} = \frac{1}{2}\frac{\left(1-r^2\right)^2}{r^2}\frac{g_{\text{rt}}}{\left(1-g_{\text{rt}}\right)^2}$$

$$= \frac{\left(1-r^2\right)^2}{r^2}\frac{g_{\text{rt}}}{1+g_{\text{rt}}^2-2g_{\text{rt}}\cos\left(2m\pi+\delta\phi\right)}$$

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$\delta\omega$

$$1+g_{\text{rt}}^2-2g_{\text{rt}}\cos(2m\pi+\delta\phi) = 2\left(1-g_{\text{rt}}\right)^2$$

$$1+g_{\text{rt}}^2-2g_{\text{rt}}\left[1-\frac{1}{2}\left(\delta\phi\right)^2\right] \approx 2\left(1-g_{\text{rt}}\right)^2 \Rightarrow$$

$$g_{\text{rt}}\left(\delta\phi\right)^2 = \left(1-g_{\text{rt}}\right)^2, \quad \delta\phi = \frac{1-g_{\text{rt}}}{\sqrt{g_{\text{rt}}}}$$

$$\frac{\delta\phi}{2m\pi} = \frac{\delta\omega}{\omega}, \quad \frac{\delta\phi}{2\pi} = \frac{\delta\omega}{m\omega_c}m = \frac{\delta\omega}{\omega_c}, \quad \omega_c = 2\pi\frac{c}{2L}$$

$$2\delta\omega = \Delta\omega_{\text{fwhm}} = 2\times 2\pi\frac{c}{2L}\frac{\delta\phi}{2\pi} = \frac{c}{L}\frac{1-g_{\text{rt}}}{\sqrt{g_{\text{rt}}}}$$

$$\approx \frac{c}{L}\left(\delta_0+\delta_r\right)$$

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Project

- Progress discussion: **2 September 2019**.
- Presentation and Report submission: **23 September 2019**