RATE EQUATIONS

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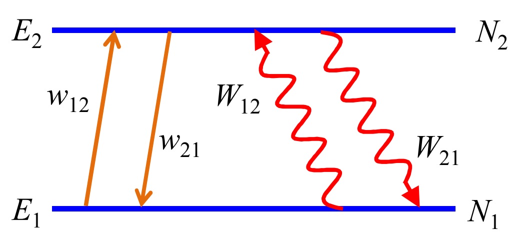
Two-Level System



$$\frac{dN_1(t)}{dt} = -\left[W_{12} + w_{12}\right]N_1(t) + \left[W_{21} + w_{21}\right]N_2(t)$$

$$\frac{dN_2(t)}{dt} = \left[W_{12} + w_{12}\right]N_1(t) - \left[W_{21} + w_{21}\right]N_2(t)$$

$$N_1(t) + N_2(t) = N, \qquad N_1(t) - N_2(t) = \Delta N$$

$$N_1(t) = \frac{N + \Delta N}{2}, \qquad N_2(t) = \frac{N - \Delta N}{2}$$

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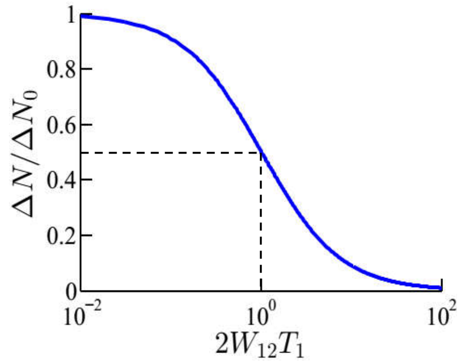
Saturable Absorption, Gain

$$\alpha_m = \alpha_{m0} \frac{1}{1 + I / I_{\text{sat}}}$$



- When a laser oscillator begins to oscillate, the oscillation amplitude grows at first until the intensity inside the cavity is sufficient to saturate down the laser gain.
- Steady-state oscillation then occurs when the saturated laser gain becomes just equal to the total cavity losses, so that the net round-trip gain is exactly unity.

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Time Dependence

- Suppose we switch-on the light at $t = 0$. Let's see the time dependence of the population difference.

$$\frac{d\Delta N}{dt} = -2W_{12}\Delta N - \frac{\Delta N - \Delta N_0}{T_1} = 0 \Rightarrow \frac{d\Delta N}{dt} = -\left[2W_{12} + \frac{1}{T_1}\right]\Delta N + \frac{\Delta N_0}{T_1}$$

$$\Delta N = Ae^{-\lambda t} + B \Rightarrow \frac{d\Delta N}{dt} = -\lambda A e^{-\lambda t} = -\lambda\left(\Delta N - B\right) = -\lambda \Delta N + \lambda B$$

$$\lambda = 2W_{12} + \frac{1}{T_1},\ B\left(2W_{12} + \frac{1}{T_1}\right) = \frac{\Delta N_0}{T_1} \Rightarrow B = \frac{\Delta N_0}{2W_{12}T_1 + 1}$$

$$\Delta N(0) = A + B \Rightarrow A = \Delta N(0) - B$$

$$\Delta N = \left(\Delta N(0) - \frac{\Delta N_0}{2W_{12}T_1 + 1}\right) e^{-\left(2W_{12} + \frac{1}{T_1}\right)t} + \frac{\Delta N_0}{2W_{12}T_1 + 1}$$

$$= \Delta N_{\text{ss}} + \left[\Delta N(0) - \Delta N_{\text{ss}}\right] e^{-\left(2W_{12} + \frac{1}{T_1}\right)t}$$

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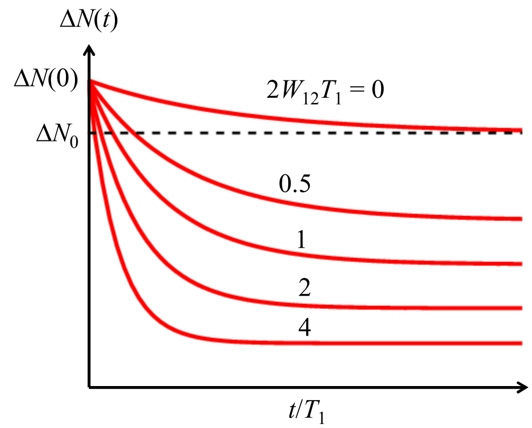
Transient

$$\Delta N = \Delta N_{ss} + [\Delta N(0) - \Delta N_{ss}]e^{-\left(2W_{12}+\frac{1}{T_1}\right)t}$$

$$\Delta N_{ss} = \Delta N_0 \frac{1}{1 + 2W_{12}T_1}$$



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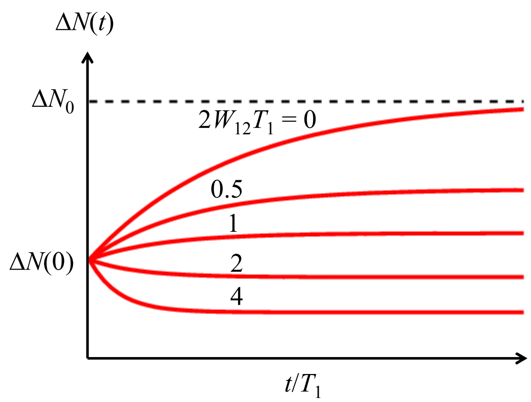
Transient

$$\Delta N = \Delta N_{ss} + [\Delta N(0) - \Delta N_{ss}]e^{-\left(2W_{12}+\frac{1}{T_1}\right)t}$$

$$\Delta N_{ss} = \Delta N_0 \frac{1}{1 + 2W_{12}T_1}$$



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Degeneracy

$$g_1 W_{12} = g_2 W_{21}$$

$$\frac{w_{12}}{w_{21}} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT}$$

$$\Delta N(t) = \frac{g_2}{g_1} N_1(t) - N_2(t)$$

$$\Delta N_0 = \frac{g_2}{g_1} N_{10} - N_{20} = N \frac{1 - e^{-\hbar\omega_a/kT}}{1 + \frac{g_1}{g_2} e^{-\hbar\omega_a/kT}}$$

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Rate Equation Approximation

Note:

- Polarization decay is not considered.
- Photon equation is not considered.

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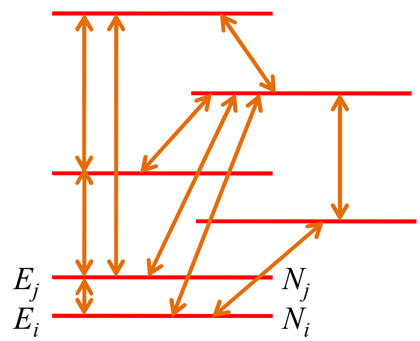
Multilevel Rate Equations



$$\frac{dN_i}{dt} = \sum_{j \neq i} \left(-W_{ij} N_i + W_{ji} N_j \right) + \sum_{j \neq i} \left(-w_{ij} N_i + w_{ji} N_j \right)$$

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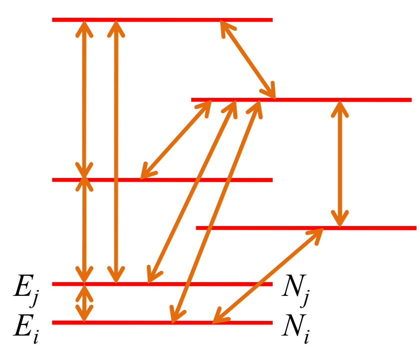
Conservation



- If the total number of atoms in all the energy levels is constant:

$$\sum_{i=1}^{M} N_i = N_1 + N_2 + \cdots + N_M = N$$

- Then only $M - 1$ of the M rate equations are linearly independent.

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Steady-State

$$\hat{W}_{11}N_{1,\text{ss}} + \hat{W}_{21}N_{2,\text{ss}} + \cdots + \hat{W}_{M1}N_{M,\text{ss}} = 0$$
$$\hat{W}_{12}N_{1,\text{ss}} + \hat{W}_{22}N_{2,\text{ss}} + \cdots + \hat{W}_{M2}N_{M,\text{ss}} = 0$$
$$\cdots \quad + \quad \cdots \quad + \cdots + \quad \cdots \quad = 0$$
$$N_{1,\text{ss}} + \quad N_{2,\text{ss}} + \cdots + \quad N_{M,\text{ss}} = N$$

$$\hat{W}_{ij} = W_{ij} + w_{ij}$$

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Optical Frequency

Upward relaxation rates can be ignored.

$$w_{ij} \approx \gamma_{ij}$$

Rate equations →

$$\frac{dN_i}{dt} = -\sum_{k<i} \gamma_{ik} N_i + \sum_{k>i} \gamma_{ki} N_k = -\gamma_i N_i = -\frac{N_i}{\tau_i}$$

The total decay rate →

$$\gamma_i = \frac{1}{\tau_i} = \sum_{k<i} \gamma_{ik}$$

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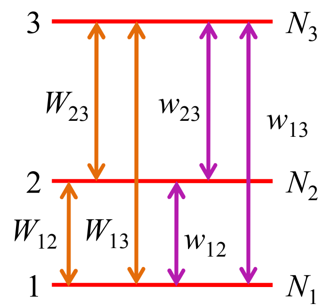
3-Level System

Assume that applied signals present on all three transitions.



$$\frac{dN_1}{dt} = -\left(W_{12} + W_{13} + w_{12} + w_{13}\right) N_1 + \left(W_{21} + w_{21}\right) N_2 + \left(W_{31} + w_{31}\right) N_3$$

$$\frac{dN_2}{dt} = -\left(W_{21} + W_{23} + w_{21} + w_{23}\right) N_2 + \left(W_{12} + w_{12}\right) N_1 + \left(W_{32} + w_{32}\right) N_3$$

$$\frac{dN_3}{dt} = -\left(W_{31} + W_{32} + w_{31} + w_{32}\right) N_3 + \left(W_{23} + w_{23}\right) N_2 + \left(W_{13} + w_{13}\right) N_1$$

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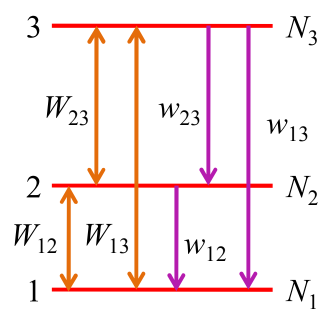
3-Level System

Optical Frequency Approximation:



$$\frac{dN_1}{dt} = -\left(W_{12} + W_{13}\right) N_1 + \left(W_{21} + w_{21}\right) N_2 + \left(W_{31} + w_{31}\right) N_3$$

$$\frac{dN_2}{dt} = -\left(W_{21} + W_{23} + w_{21}\right) N_2 + W_{12} N_1 + \left(W_{32} + w_{32}\right) N_3$$

$$\frac{dN_3}{dt} = -\left(W_{31} + W_{32} + w_{31} + w_{32}\right) N_3 + W_{23} N_2 + W_{13} N_1$$

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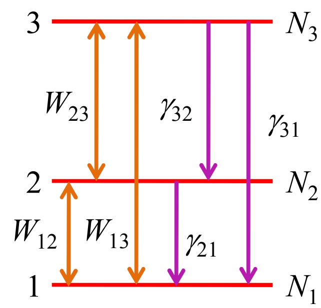
3-Level System

Optical Frequency Approximation:

$$w_{ij} \approx \gamma_{ij}$$

$$\gamma_i = \frac{1}{\tau_i} = \sum_{k<i} \gamma_{ik}$$



$$\frac{dN_1}{dt} = -\left(W_{12} + W_{13}\right)N_1 + \left(W_{21} + \gamma_{21}\right)N_2 + \left(W_{31} + \gamma_{31}\right)N_3$$

$$\frac{dN_2}{dt} = -\left(W_{21} + W_{23} + \gamma_{21}\right)N_2 + W_{12}N_1 + \left(W_{32} + \gamma_{32}\right)N_3$$

$$\frac{dN_3}{dt} = -\left(W_{31} + W_{32} + \gamma_3\right)N_3 + W_{23}N_2 + W_{13}N_1$$

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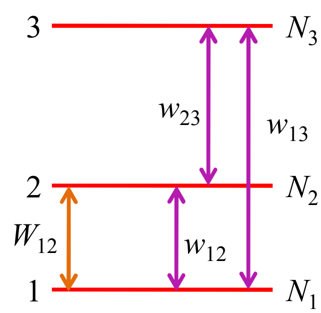
Example

- Assume that applied signal is between levels 1 and 2.
- Determine steady-state population densities.



$$\frac{dN_1}{dt} = -\left(W_{12} + w_{12} + w_{13}\right)N_1 + \left(W_{21} + w_{21}\right)N_2 + w_{31}N_3$$

$$\frac{dN_2}{dt} = -\left(W_{21} + w_{21} + w_{23}\right)N_2 + \left(W_{12} + w_{12}\right)N_1 + w_{32}N_3$$

$$\frac{dN_3}{dt} = -\left(w_{31} + w_{32}\right)N_3 + w_{23}N_2 + w_{13}N_1$$

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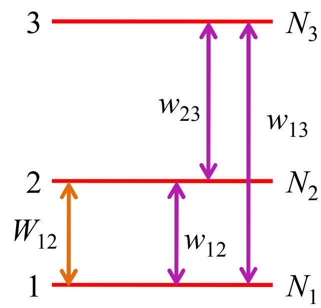
Example

- **Steady-state**



$$\frac{dN_1}{dt}=0 \Rightarrow -\left(W_{12}+w_{12}+w_{13}\right)N_{1,ss}+\left(W_{21}+w_{21}\right)N_{2,ss}+w_{31}N_{3,ss}=0$$

$$\frac{dN_2}{dt}=0 \Rightarrow \left(W_{12}+w_{12}\right)N_{1,ss}-\left(W_{21}+w_{21}+w_{23}\right)N_{2,ss}+w_{32}N_{3,,ss}=0$$

$$\frac{dN_3}{dt}=0 \Rightarrow w_{13}N_{1,ss}+w_{23}N_{2,ss}-\left(w_{31}+w_{32}\right)N_{3,ss}=0$$

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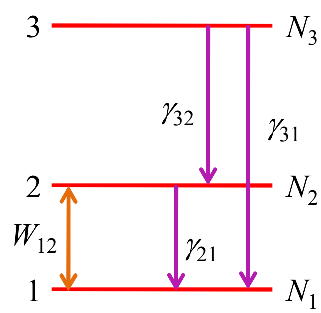
Example

- **Optical frequency approximation**



$$-W_{12}N_{1,ss}+\left(W_{21}+\gamma_{21}\right)N_{2,ss}+\gamma_{31}N_{3,ss}=0$$

$$W_{12}N_{1,ss}-\left(W_{21}+\gamma_{21}\right)N_{2,ss}+\gamma_{32}N_{3,,ss}=0$$

$$-\gamma_3 N_{3,ss}=0$$

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FOUR-LEVEL SYSTEM

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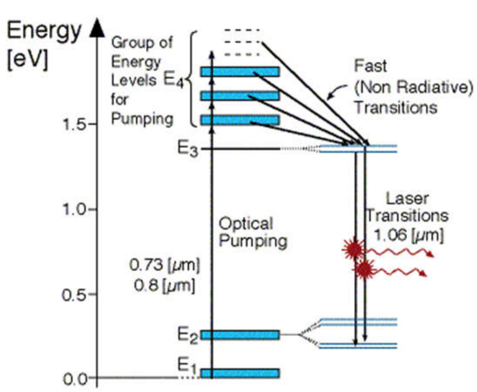
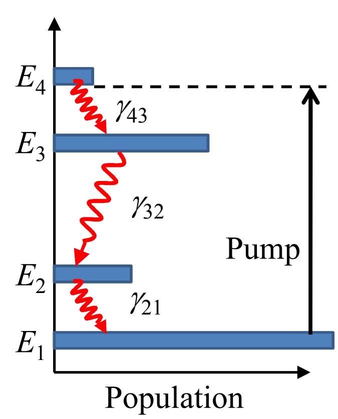
Nd:YAG Lasers

- Energy diagram of Nd:YAG laser system





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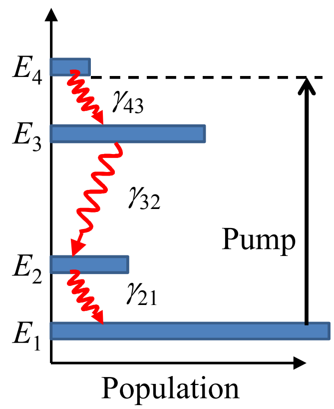
Four-Level System

- **Level 4:** Combination of all the levels lying above the upper laser level.
- **Level 3:** Upper laser level, long-lived level
- **Level 2:** Lower laser level, short-lived level
- **Level 1:** Ground level
- **Other levels:** Ignore
- **Example:** Nd:YAG lasers



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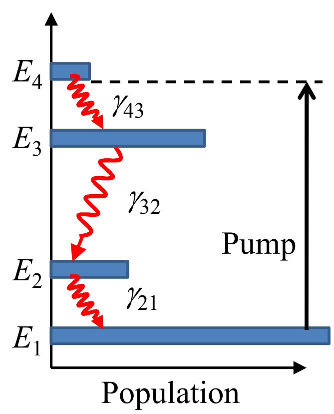
Transition Rates

- γ_{43}: Fast
- γ_{32} : Slow
- γ_{21} : Fast



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Pumping

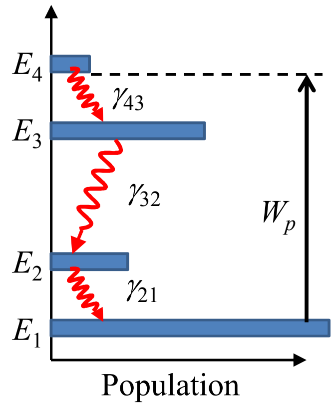
Consider optical frequency approximation.

$$W_{14} = W_{41} = W_p$$

$$\frac{dN_4}{dt} = W_p\left(N_1 - N_4\right) - \left(\gamma_{43} + \gamma_{42} + \gamma_{41}\right)N_4$$

$$= W_p(N_1 - N_4) - \frac{N_4}{\tau_4}$$

$$\frac{1}{\tau_4} = \gamma_4 = \gamma_{43} + \gamma_{42} + \gamma_{41}$$



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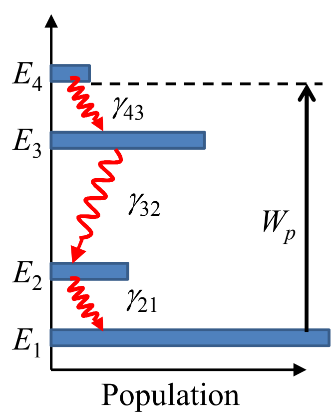
Steady-State

$$\frac{dN_4}{dt} = 0 \Rightarrow W_p(N_1 - N_4) - \frac{N_4}{\tau_4} = 0$$

$$W_p N_1 - \left(W_p + \frac{1}{\tau_4}\right)N_4 = 0$$

$$N_4 = \frac{W_p\tau_4}{1 + W_p\tau_4}N_1$$

$$\approx W_p\tau_4 N_1 \quad \text{if } W_p\tau_4 \ll 1$$



- Usually $W_p\tau_4 \ll 1$, so $N_4 \ll N_1$.
- $1 \rightarrow 3$ transition is negligible.

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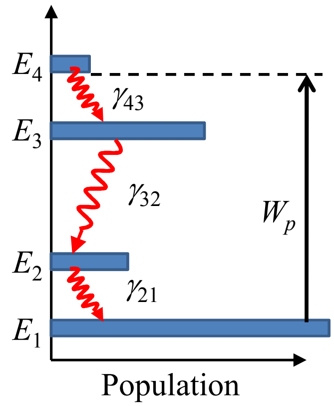
Lasing Levels

$$\frac{dN_3}{dt} = \gamma_{43}N_4 - (\gamma_{32} + \gamma_{31})N_3$$
$$= \frac{N_4}{\tau_{43}} - \frac{N_3}{\tau_3}$$

$$\frac{dN_2}{dt} = \gamma_{42}N_4 + \gamma_{32}N_3 - \gamma_{21}N_2$$
$$= \frac{N_4}{\tau_{42}} + \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}}$$



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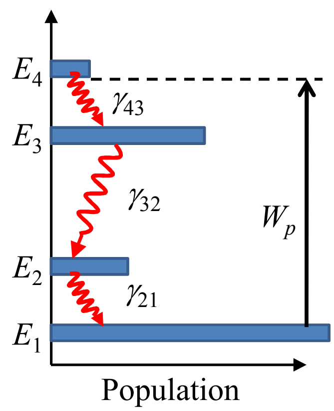
Steady-State

$$\frac{dN_3}{dt} = 0 \Rightarrow \frac{N_4}{\tau_{43}} - \frac{N_3}{\tau_3} = 0$$

$$N_3 = \frac{\tau_3}{\tau_{43}}N_4$$

In a good laser $\tau_3 >> \tau_{43}$ and hence $N_3 >> N_4$.



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Steady-State

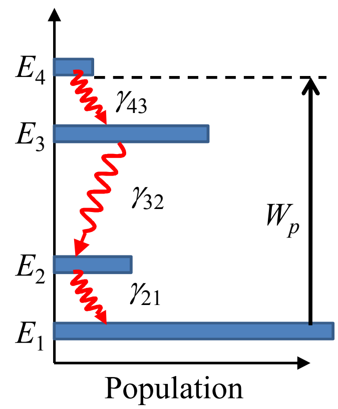
$$\frac{dN_2}{dt} = 0 \Rightarrow \frac{N_4}{\tau_{42}} + \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}} = 0$$

$$\frac{\tau_{43} N_3}{\tau_3 \tau_{42}} + \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}} = 0$$

$$N_2 = \left(\frac{\tau_{21}}{\tau_{32}} + \frac{\tau_{43}\tau_{21}}{\tau_3 \tau_{42}} \right) N_3 = \beta N_3$$

In a good laser $\tau_{42} \approx \infty$.

$$\beta = \frac{N_2}{N_3} \approx \frac{\tau_{21}}{\tau_{32}} \gg 1$$



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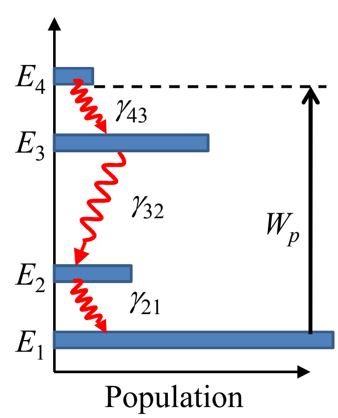
Quantum Efficiency

Number of fluorescent photons spontaneously emitted on the laser transition divided by the number of pump photons absorbed.

$$\eta = \frac{\gamma_{43}}{\gamma_4} \times \frac{\gamma_{rad}}{\gamma_3} = \frac{\tau_4}{\tau_{43}} \times \frac{\tau_3}{\tau_{rad}}$$

$$\gamma_{rad} = \gamma_{rad}(3 \rightarrow 2)$$



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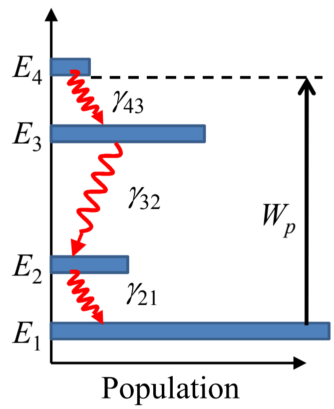
Population Inversion

$$N_1 + N_2 + N_3 + N_4 = N$$

$$\frac{N_3 - N_2}{N} = \frac{(1-\beta)\eta W_p \tau_{\text{rad}}}{1 + \left[1 + \beta + 2\frac{\tau_{43}}{\tau_{\text{rad}}}\right]\eta W_p \tau_{\text{rad}}}$$

In a good laser, $\tau_{43} \ll \tau_{\text{rad}}$.

$$\frac{N_3 - N_2}{N} = \frac{(1-\beta)\eta W_p \tau_{\text{rad}}}{1 + (1+\beta)\eta W_p \tau_{\text{rad}}}$$



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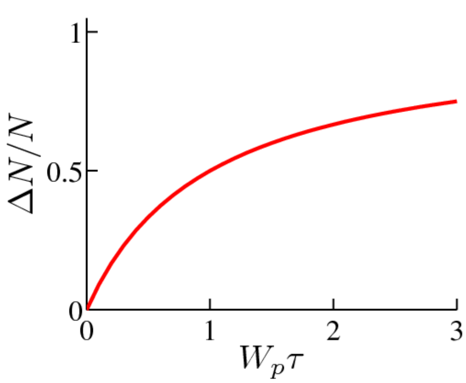
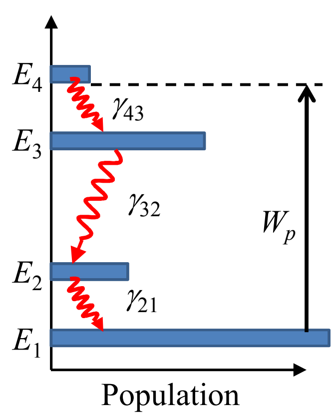
Population Inversion

If $\beta \approx \frac{\tau_{21}}{\tau_{32}} \to 0$

$$\frac{N_3 - N_2}{N} = \frac{W_p \tau_{\text{rad}}}{1 + W_p \tau_{\text{rad}}}$$





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THREE-LEVEL SYSTEM

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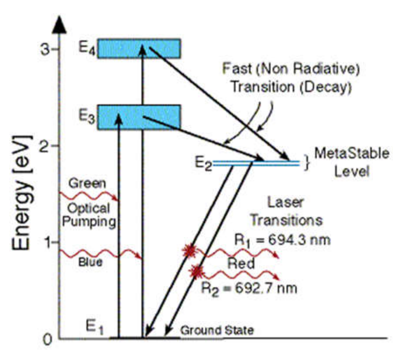
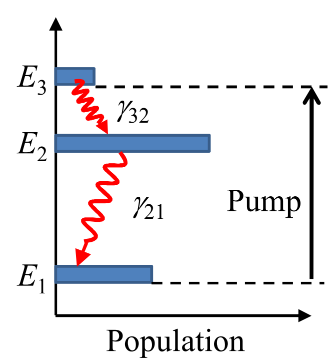
Ruby Laser

- Energy diagram of Ruby laser:





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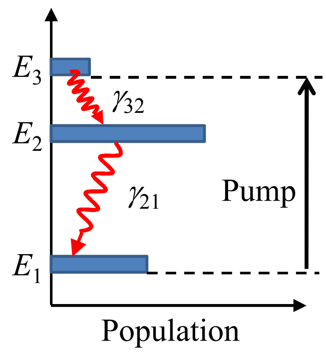
Three-Level System

- **Level 3:** Combination of all the levels lying above the upper laser level.
- **Level 2:** Upper laser level, long-lived level
- **Level 1:** Lower laser level, also the ground level
- **Other levels:** Ignore
- **Example:** Ruby lasers
- Usually is not as efficient as four-level system



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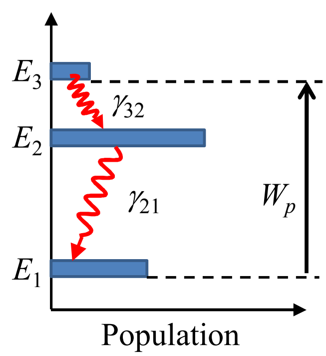
Transition Rates

- γ_{32}: Fast
- γ_{21} : Slow



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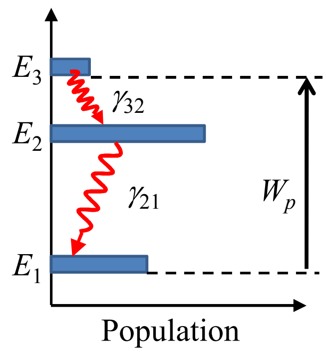
Rate Equations

Consider optical frequency approximation.

$$W_{13} = W_{31} = W_p$$

$$\frac{dN_3}{dt} = W_p\left(N_1 - N_3\right) - \frac{N_3}{\tau_3}$$

$$\frac{dN_2}{dt} = \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}}$$



Conservation of atoms:

$$N_1 + N_2 + N_3 = N$$

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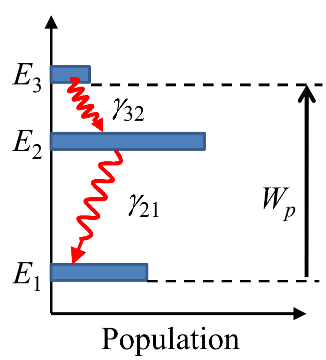
Quantum Efficiency

$$\eta = \frac{\gamma_{32}}{\gamma_3} \times \frac{\gamma_{\text{rad}}}{\gamma_{21}} = \frac{\tau_3}{\tau_{32}} \times \frac{\tau_{21}}{\tau_{\text{rad}}}$$

$$\gamma_{\text{rad}} = \gamma_{\text{rad}}(2 \to 1)$$



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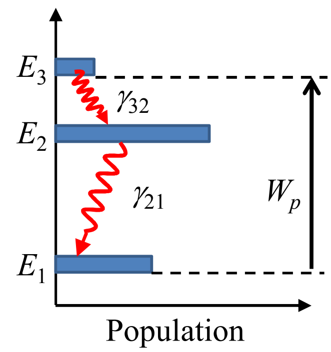
Population Inversion

$$\frac{N_2 - N_1}{N} = \frac{(1-\beta)\eta W_p \tau_{\text{rad}} - 1}{(1+2\beta)\eta W_p \tau_{\text{rad}} + 1}$$

$$\beta = \frac{N_3}{N_2} = \frac{\tau_{32}}{\tau_{21}}$$

Inversion can be obtained only if $\beta < 1$.



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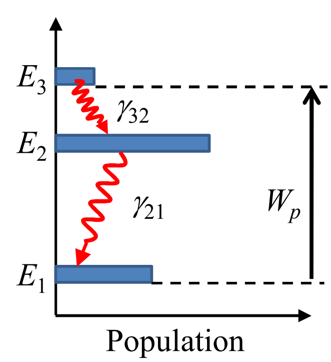
Population Inversion

Pumping threshold

$$W_p \tau_{\text{rad}} \geq \frac{1}{\eta(1-\beta)}$$

if $\eta \to 1$ and $\beta \to 0$

$$\frac{N_2 - N_1}{N} \approx \frac{W_p \tau_{\text{rad}} - 1}{W_p \tau_{\text{rad}} + 1}$$



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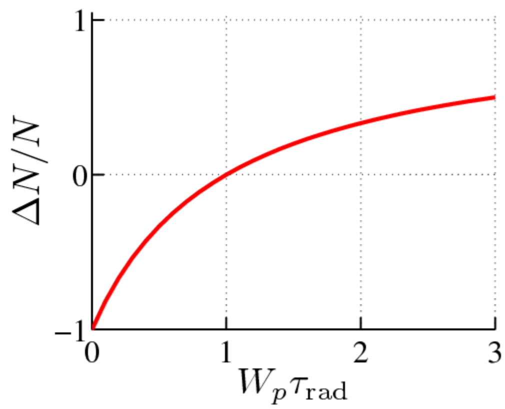
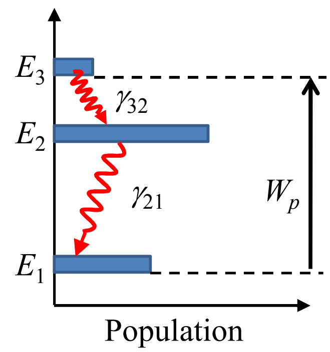
Population Inversion

$$\frac{N_2 - N_1}{N} \approx \frac{W_p \tau_{\text{rad}} - 1}{W_p \tau_{\text{rad}} + 1}$$





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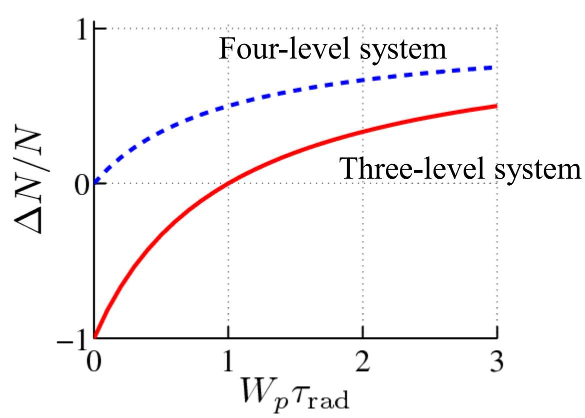
Comparison



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Project

- Proposal submission: **29 July 2019**.
- Send your tentative title, and your and your partner's name and student numbers by **27 July 2019**.