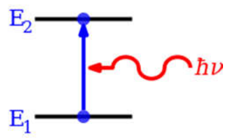
POPULATION INVERSION

1

Condition for Gain

- Energy transfer rate to the atoms

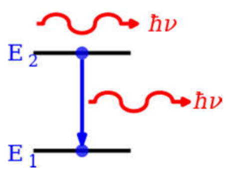
$$\frac{dU_a}{dt} = Kn(t)\left[N_1(t) - N_2(t)\right]\hbar\omega = -\frac{dU_{\text{sig}}}{dt}$$



- Energy transfer rate to the signal

$$U_{\text{sig}} = n(t)\hbar\omega$$

$$\frac{dn(t)}{dt} = -K\left[N_1(t) - N_2(t)\right]n(t)$$



- $n(t)$ may either decay or grow with time, depending on the sign of $\Delta N(t) = N_1(t) - N_2(t)$.

Population Inversion $\rightarrow N_2(t) > N_1(t)$

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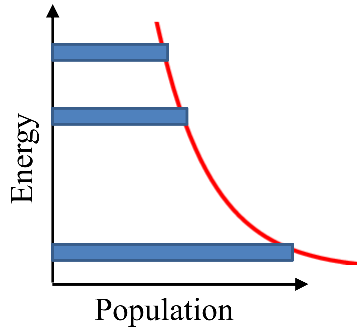
Equilibrium

Boltzmann's Principle

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{kT}\right)$$

$$\Delta N = N_1 - N_2 = N_1\left(1 - e^{-\hbar\omega/kT}\right)$$

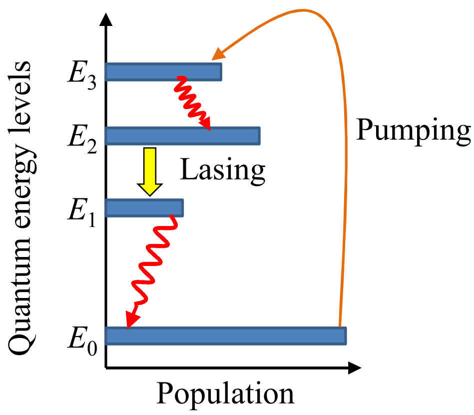
$$N_1 \gg N_2$$



We must find some pumping process for gain → create non-equilibrium condition.

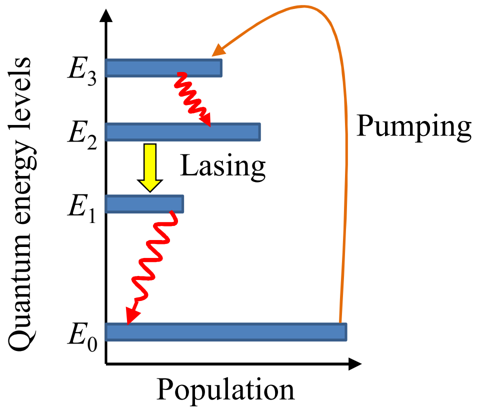
Pumping

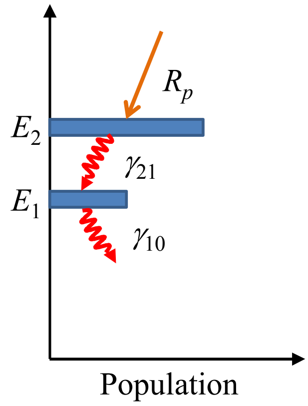
4 level pumping model



Pumping

4 level pumping model





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Rate Equations

$$\frac{dN_2}{dt} = R_p - \gamma_{21} N_2$$

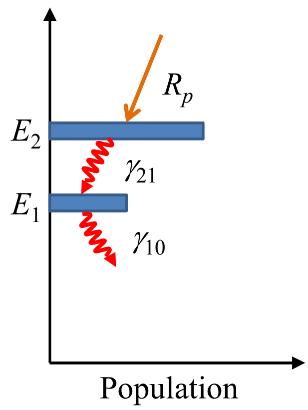
$$\frac{dN_1}{dt} = \gamma_{21} N_2 - \gamma_{10} N_1$$

At equilibrium:

$$\frac{dN_2}{dt} = 0 \rightarrow N_{2,ss} = R_p / \gamma_{21}$$

$$\frac{dN_1}{dt} = 0 \rightarrow N_{1,ss} = \left(\gamma_{21} / \gamma_{10}\right) N_{2,ss}$$

$$\left(N_2 - N_1\right)_{ss} = \frac{R_p\left(\gamma_{10} - \gamma_{21}\right)}{\gamma_{10}\gamma_{21}} = R_p \tau_{21}\left(1 - \frac{\tau_{10}}{\tau_{21}}\right)$$



If $\gamma_{10} > \gamma_{21}$, so that $\tau_{10} < \tau_{21}$, we will have population inversion.

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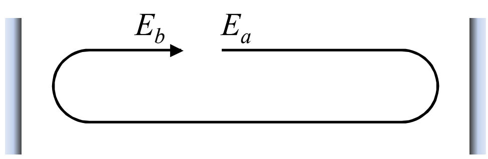
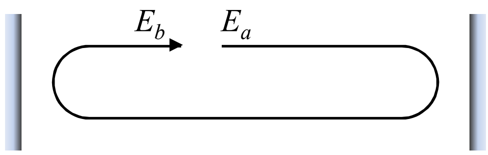
LASER OSCILLATION

Laser Oscillation



The noise radiation originated from the spontaneous emission exponentially grows and leads to a coherent self-sustained oscillation inside the cavity if net gain is greater than the net loss.

Steady State Oscillation



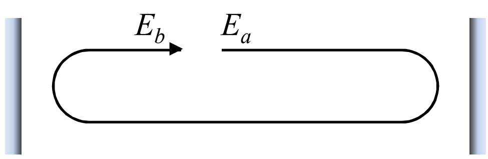
Mirror reflectivity

Change in amplitude due to gain and loss

$$\frac{E_b}{E_a} = r_1 r_2\, e^{(g-\alpha)2L}\, e^{-jk2L} = 1$$

Phase accumulation due to propagation

Oscillation Conditions



Gain:

$$r_1 r_2\, e^{(g-\alpha)2L} = 1$$

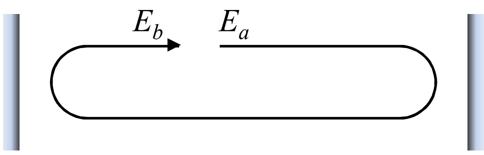
$$g = \alpha + \frac{1}{2L}\ln\frac{1}{r_1 r_2}$$

Frequency:

$$2kL = m2\pi \Rightarrow 2n\frac{\omega}{c}L = m2\pi$$

$$2n\frac{2\pi f}{c}L = m2\pi \Rightarrow f = m\frac{c}{2nL}$$

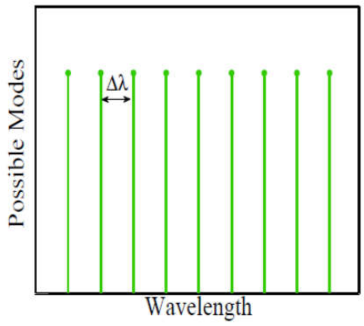
Modes

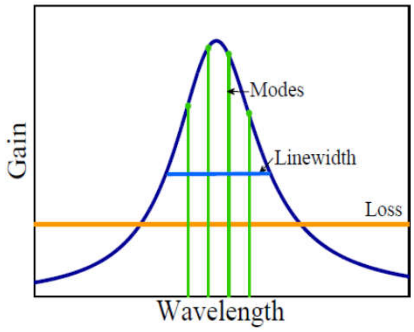


$$f_m = m\frac{c}{2nL}$$

- f_m are the frequencies of the allowed longitudinal modes.
- The modes are spaced in frequency by $c/(2nL)$. If L = 1 m and n = 1.5, the mode spacing is 100 MHz.

Modes





- More than one mode can exist if linewidth of transitions is greater than the mode spacing.

STIMULATED TRANSITIONS: CLASSICAL OSCILLATOR MODEL

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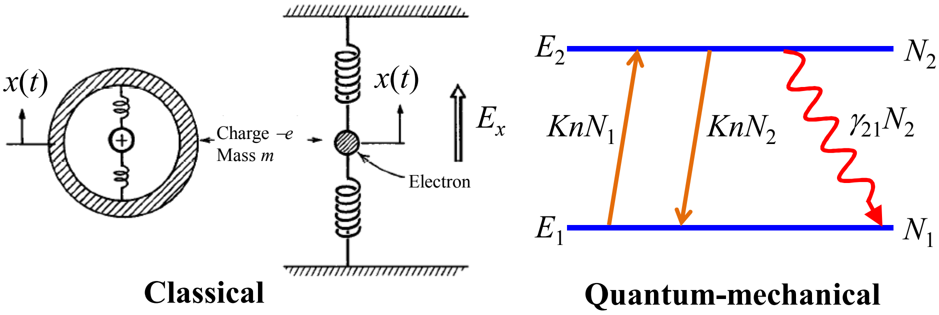
Classical Oscillator Model

- Interactions: **optical signals ↔ atoms**



- Instantaneous displacement : $x(t)$
- Restoring force: $-Kx(t)$
- Externally applied field: $E_x(t)$

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Why Electric Force is Stronger

We will use a plane electro-magnetic wave approximation.

Lorentz Force Equation:

$$\vec{F} = \vec{F}_E + \vec{F}_B$$

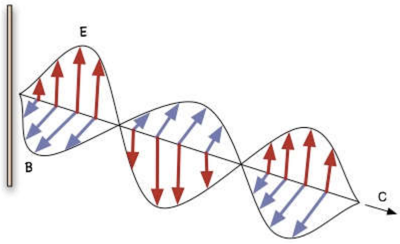
$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$E = E_0 e^{j(\omega t - kz)},\; B = B_0 e^{j(\omega t - kz)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$kE_0 = \omega B_0,\; B_0 = \frac{k}{\omega} E_0 = \frac{E_0}{c}$$

$$\frac{F_B}{F_E} = \frac{v}{c} < 1$$



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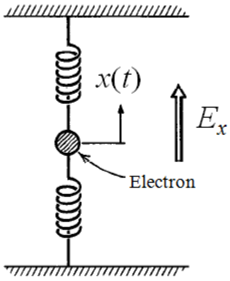
Equation of Motion

$$m\frac{d^2x(t)}{dt^2} = -Kx(t) - eE_x(t)$$

$$\frac{d^2x(t)}{dt^2} = -\frac{K}{m}x(t) - \frac{e}{m}E_x(t)$$

$$\frac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = -\frac{e}{m}E_x(t)$$



ω_0 : CEO resonance frequency

Quantum-mechanics

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

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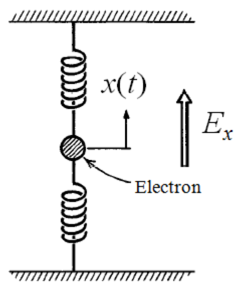
Damping

- *We have a never-ending oscillation without an electric field!*

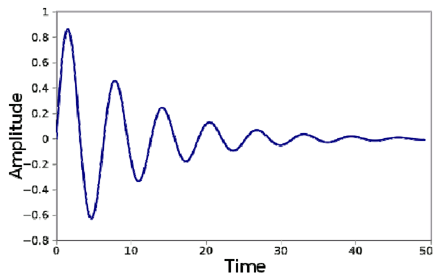
  → We need to add damping.



$$\frac{d^2x(t)}{dt^2}+\gamma\frac{dx(t)}{dt}+\omega_0^2x(t)=-\frac{e}{m}E_x(t)$$

γ: Damping rate

$\omega_0 \gg \gamma$



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Solution with no Excitation

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2x=0, x=e^{at}\Rightarrow a^2+\gamma a+\omega_0^2=0$$

$$a=\frac{-\gamma\pm\sqrt{\gamma^2-4\omega_0^2}}{2}=\frac{-\gamma\pm j\sqrt{4\omega_0^2-\gamma^2}}{2}$$

$$a=-\frac{\gamma}{2}\pm j\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}$$ → Consider only positive frequency dependency

$$x=x_0e^{-\frac{\gamma}{2}t}e^{j\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}t}$$ → A damping harmonic oscillator.

The damping shifts the oscillation frequency slightly from ω_0.

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Time Averaged Energy

- If we average over the cycle of oscillation to compute an energy loss rate of the oscillating atoms:

$$\begin{aligned}\langle U\rangle &= \langle P.E.\rangle_{\text{avg.}} + \langle K.E.\rangle_{\text{avg.}} \\ &= \frac{1}{2}k\langle x^2\rangle_{\text{avg.}} + \frac{1}{2}m\left\langle \frac{dx}{dt}\right\rangle_{\text{avg.}} \\ &= \langle U_0\rangle e^{-\gamma t}\end{aligned}$$

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Energy Loss

$$\gamma = \left|\frac{1}{\langle U\rangle}\frac{d\langle U\rangle}{dt}\right| = \gamma_{\text{rad}} + \gamma_{\text{nr}}$$

- Here γ is the energy loss rate which we have explicitly broken out into radiative and non-radiative terms.
- The radiative part is due to spontaneous emission.
- The non-radiative energy loss term is due to inelastic collisions with other atoms, walls, etc. In solids this loss is due to a coupling of the energy into the lattice.

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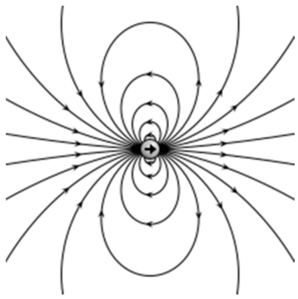
Dipole Moment

- Dipole moment of an individual atom

$$\mu_x(t) = [\text{charge}] \times [\text{displacement}] = -ex(t)$$



Microscopic to Macroscopic

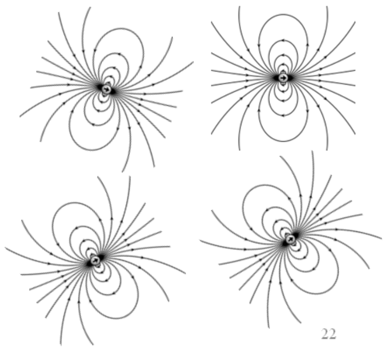
- In a collection of atoms, we must sum over all of the dipoles to get a collective response.
- We have to take an average over dipoles in a volume.
- The volume should be large enough so that it contains a large number of atoms but small enough so that the atoms see the same optical phase from and electromagnetic field.

$$\vec{\mu}_{x_i}(t) = -e\vec{x}_i(t)$$

$$\vec{\mathrm{p}} = \sum_{i=1}^{NV} \vec{\mu}_{x_i}(t)/V$$

V : Volume

N : Density of dipoles

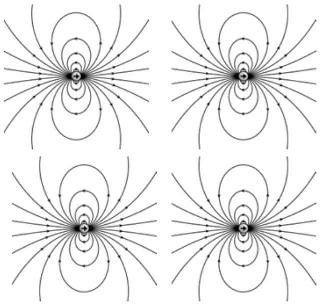


Coherent Dipole Oscillations

- Single microscopic dielectric oscillator

$$-ex = -ex_0 e^{-\frac{\gamma}{2}t} e^{j\sqrt{\omega_0^2-(\gamma/2)^2}\,t}$$

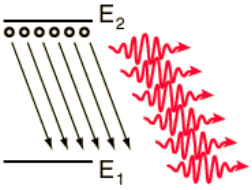
$$\mu_{x_i}(t) = \mu_{x_{i0}} e^{-\frac{\gamma}{2}t} e^{j\sqrt{\omega_0^2-(\gamma/2)^2}\,t} e^{j\phi}$$



- If the dipoles oscillate in phase

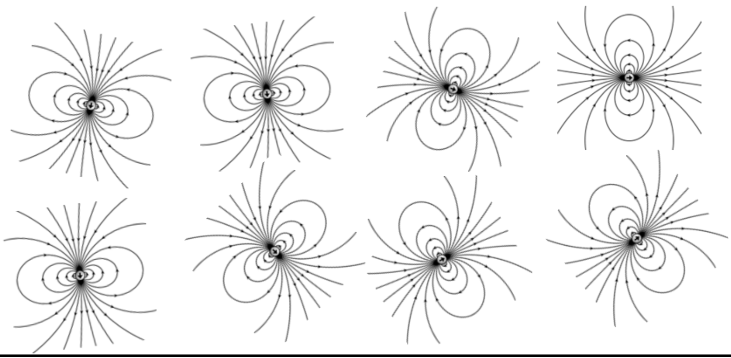
$$p = \sum_{i=1}^{N\Delta V} \mu_{x_i} / \Delta V = \frac{N\Delta V \vec{\mu}_{x_i}}{\Delta V}$$

$$= N\mu_{x_i} e^{-\frac{\gamma}{2}t} e^{j\sqrt{\omega_0^2-(\gamma/2)^2}\,t} e^{j\phi}$$



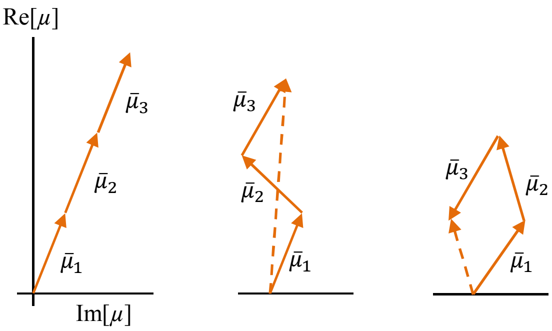
Collisions and Dephasing

- When we drive the dipoles with a field they should all be in phase (in the small volume element).
- What we know is that collisions with other atoms, etc. cause the phase of the oscillator to be perturbed.
- This is a quantum-mechanical phenomenon, but we can use classical oscillator ideas with random phases.



Dephasing

- Time phases are randomized by different scattering processes.
- Polarization becomes much smaller.
- For randomly phased dipole moments → $\langle \mu_{x,\text{tot}}(t) \rangle = 0$



Dephasing Time

- Initially, all the dipoles are oscillating in phase

$$p_{x_0} = N_0 \mu_{x_0}$$

- At $t > 0$, we have a decreasing number of dipoles that have not suffered collisions.

$$p_x(t) = N(t)\mu_x(t)$$

- If collisions occur at a random rate of $1/T_2$ collisions per atom per second. Then the decay of uncollided atoms $N(t)$ is given by

$$\frac{dN(t)}{dt} = -\frac{N(t)}{T_2} \Rightarrow N(t) = N_0 e^{-\frac{t}{T_2}}$$

Polarization

- Now assume we can align all of the dipoles in our unit volume at $t = 0$ and let them oscillate (at $t = 0$, let the external field go to zero). We find that p as a function of time

$$
\begin{aligned}
p(t) &= p_0 e^{-\frac{\gamma}{2}t} e^{j\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}\, t} e^{-\frac{t}{T_2}} \\
&= p_0 e^{-\left(\frac{\gamma}{2}+\frac{1}{T_2}\right)t + j\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}\, t}
\end{aligned}
$$

γ: Energy loss rate

$1/T_2$: Dephasing rate