CONTINUOUS CHARGE DISTRIBUTION

Disk of Charge

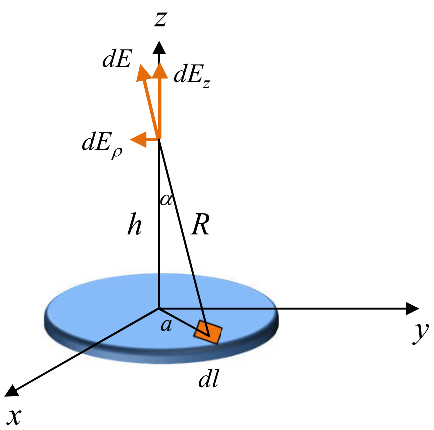
From Coulomb's law:

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Distance and Direction:

$$\vec{R} = \rho\left(-\hat{a}_\rho\right) + h\,\hat{a}_z$$

$$d\vec{E} = \frac{\rho_s \rho\, d\phi\, d\rho\left[-\rho\,\hat{a}_\rho + h\,\hat{a}_z\right]}{4\pi\varepsilon_0\left[\rho^2 + h^2\right]^{3/2}}$$



Symmetry:

$\hat{a}_\rho$ quantities cancel.

Disk of Charge

Integrate:

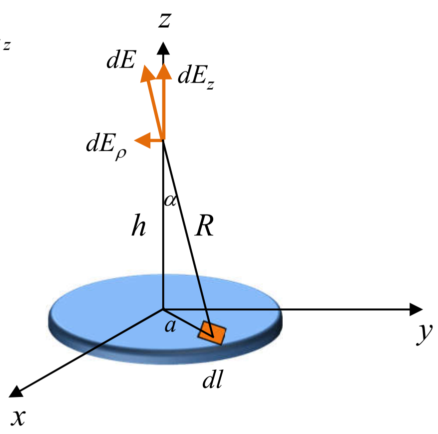
$$\vec{E} = \int d\vec{E}_z = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{h\rho\,d\rho\,d\phi}{\left[\rho^2+h^2\right]^{3/2}}\hat{a}_z$$

$$= \frac{\rho_s h}{4\pi\varepsilon_0}2\pi\int_0^a\left[\rho^2+h^2\right]^{-3/2}\frac{1}{2}d\left(\rho^2\right)\hat{a}_z$$

$$= \frac{\rho_s h}{2\varepsilon_0}\left\{-\left[\rho^2+h^2\right]^{-1/2}\right\}_0^a\hat{a}_z$$

$$= \frac{\rho_s h}{2\varepsilon_0}\hat{a}_z\left\{-\left[a^2+h^2\right]^{-1/2}+h^{-1}\right\}$$

$$= \frac{\rho_s}{2\varepsilon_0}\hat{a}_z\left\{1-\frac{h}{\left[a^2+h^2\right]^{1/2}}\right\}$$



Volume Charge

Total charge:

$$dQ = \rho_v\,dv$$

$$Q = \int \rho_v\,dv = \rho_v\int dv$$

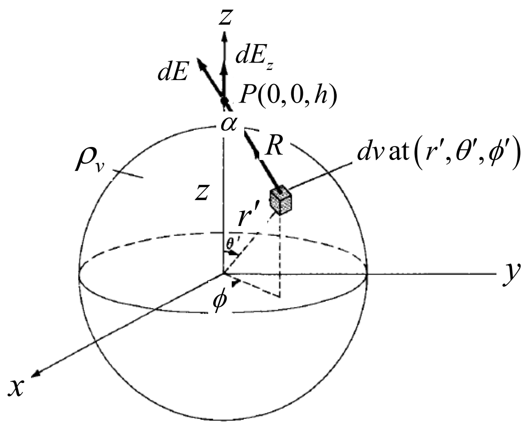
$$= \rho_v\frac{4\pi a^3}{3}$$

Coulomb's law:

$$d\vec{E} = \frac{\rho_v\,dv}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Direction:

$$\hat{a}_R = \cos\alpha\,\hat{a}_z + \sin\alpha\,\hat{a}_\rho$$



Volume Charge

Symmetry:

E_x and E_y components add up to zero.

Integrate:

$$E_z = \int dE \cos\alpha = \frac{\rho_v}{4\pi\varepsilon_0}\int \frac{dv \cos\alpha}{R^2}$$

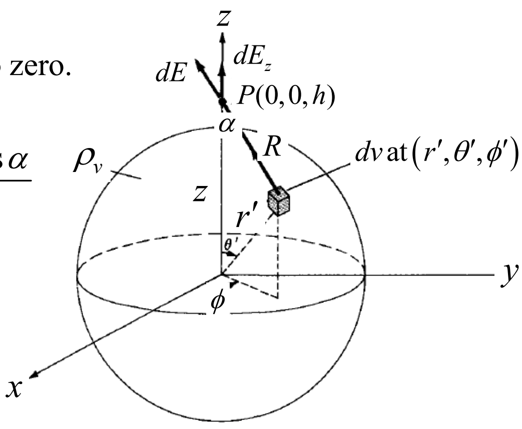
$$dv = r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR} \qquad \sin\theta' d\theta' = \frac{R\,dR}{zr'}$$



Volume Charge

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2 \frac{R\,dR}{zr'} dr' \frac{z^2 + R^2 - r'^2}{2zR}\frac{1}{R^2}$$

$$= \frac{\rho_v 2\pi}{8\pi\varepsilon_0 z^2}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\left[1 + \frac{z^2 - r'^2}{R^2}\right] dR\,dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2}\int_0^a r'\left[R - \frac{\left(z^2 - r'^2\right)}{R}\right]_{z-r'}^{z+r'} dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2}\int_0^a 4r'^2\,dr' = \frac{1}{4\pi\varepsilon_0 z^2}\left(\frac{4}{3}\pi a^3 \rho_v\right)$$

$$\vec{\mathrm{E}} = \frac{Q}{4\pi\varepsilon_0 z^2}\hat{\mathrm{a}}_z$$

GAUSS'S LAW

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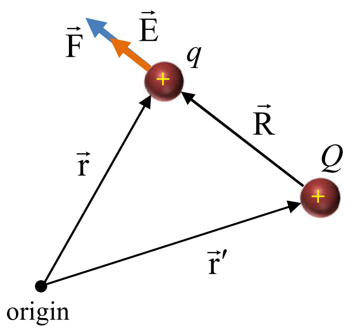
Electric Field

$$\vec{E} = \frac{\vec{F}}{q}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$= \frac{Q(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|^3}$$



Electric field intensity depends on the medium!

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Electric Flux Density

We introduce a new vector field $\vec{D}$ independent of medium.

$$\vec{D} = \varepsilon_0 \vec{E}$$

So, electric flux, $\psi = \int \vec{D} \cdot d\vec{S}$

In SI units, one line of electric flux emanates from +1C and terminates on −1C. We will learn more on flux lines later.

$\vec{D}$: Flux density → C/m^2

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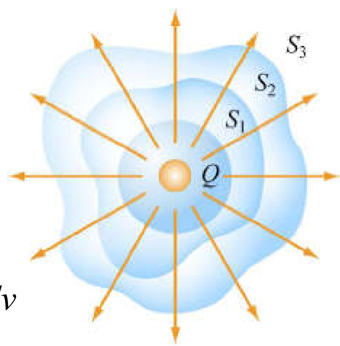
Gauss's Law – The Idea

- The total electric flux through any closed surface is equal to the total charge enclosed by that surface.



$$\psi = \oint d\psi = \oint_S \vec{D} \cdot d\vec{S}$$

$$= \text{Total charge enclosed } Q = \int_v \rho_v \, dv$$

- A very useful computational technique to find the electric field E when the source has 'enough symmetry.'

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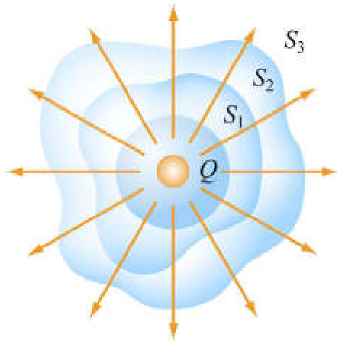
First Maxwell's Equation

$$\psi = \oint d\psi = \oint_S \vec{\mathrm{D}} \cdot d\vec{\mathrm{S}}$$

$$= \text{Total charge enclosed } Q = \int_v \rho_v \, dv$$



By applying divergence theorem

$$\oint_S \vec{\mathrm{D}} \cdot d\vec{\mathrm{S}} = \int_v \nabla \cdot \vec{\mathrm{D}} \, dv$$

Therefore

$$\boxed{\rho_v = \nabla \cdot \vec{\mathrm{D}}}$$ **→ First Maxwell's Equation**

The volume charge density is the same as the divergence of the electric flux density.

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