VECTOR CALCULUS

Divergence

$$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\cdot d\vec{S}}{\Delta v}$$

$$\oint_S \vec{A}\cdot d\vec{S} = \int_v \nabla\cdot\vec{A}\,dv$$

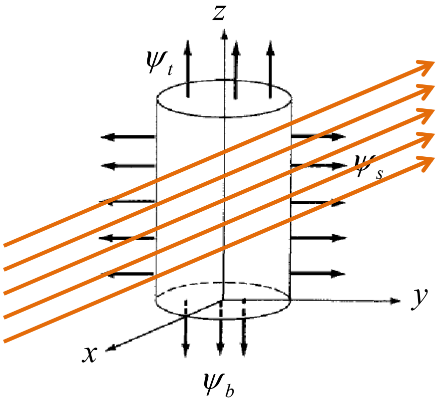
$\nabla\cdot\vec{A}$:

Commonly misinterpreted as the dot product of ∇ and $\vec{A}$.

Example

$$\rho = 1, 0 \le z \le 1$$

$$\vec{G} = 10e^{-2z}(\rho \hat{a}_\rho + \hat{a}_z)$$



- Determine the flux of $\vec{G}$ out of the entire surface of the cylinder.
- Confirm the result using the divergence theorem.

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Example

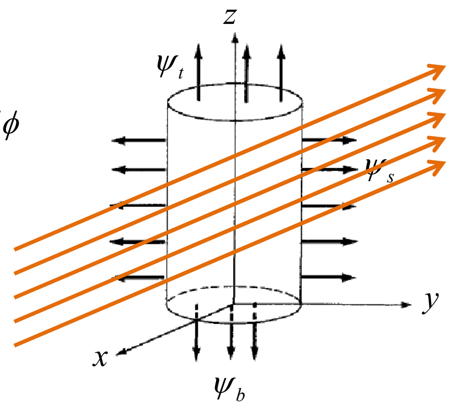
$$\psi = \oint G \cdot d\vec{S} = \psi_t + \psi_b + \psi_s$$

$$\psi_t = \int \vec{G} \cdot d\vec{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2} \rho \, d\rho \, d\phi$$

$$= 10e^{-2}(2\pi)\rho^2 / 2\Big|_0^1 = 10\pi e^{-2}$$

$$\psi_b = \int \vec{G} \cdot d\vec{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{0} \rho \, d\rho \, d\phi$$

$$= 10(2\pi)\rho^2 / 2\Big|_0^1 = -10\pi$$



$$\psi_s = \int \vec{G} \cdot d\vec{S} = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2z} \rho^2 dz \, d\phi = 10(2\pi) \frac{e^{-2z}}{-2}\Bigg|_0^1 = 10\pi(1 - e^{-2})$$

$$\psi = \psi_t + \psi_b + \psi_s = 10\pi e^{-2} - 10\pi + 10\pi(1 - e^{-2}) = 0$$

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Example

Using divergence theorem:

$$\psi = \oint_S \vec{G} \cdot d\vec{S} = \int_v (\nabla \cdot \vec{G})\, dv$$

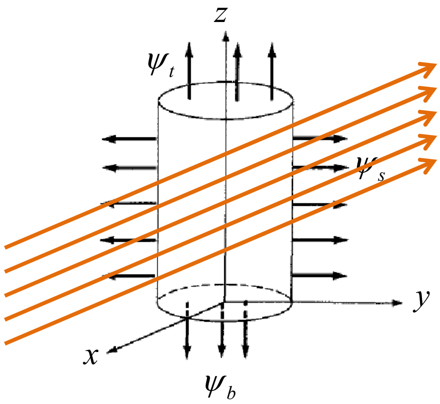
$$\nabla \cdot \vec{G}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho G_\rho\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi} G_\phi + \frac{\partial}{\partial z} G_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho^2 10 e^{-2z}\right) + \frac{\partial}{\partial z}\left(10 e^{-2z}\right)$$

$$= 20e^{-2z} - 20e^{-2z}$$

$$= 0$$



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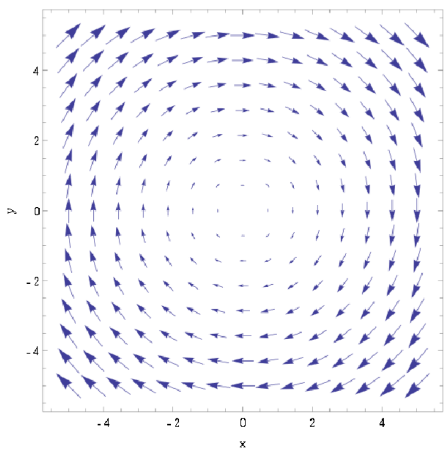
Curl of a Vector

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot d\vec{l}}{\Delta S} \right) \hat{a}_n$$



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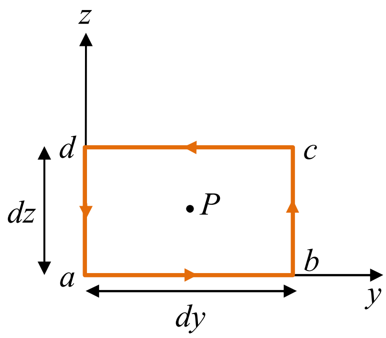
Curl of a Vector

$$\oint_L \vec{A}\cdot d\vec{l} = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da}\right)\vec{A}\cdot d\vec{l}$$



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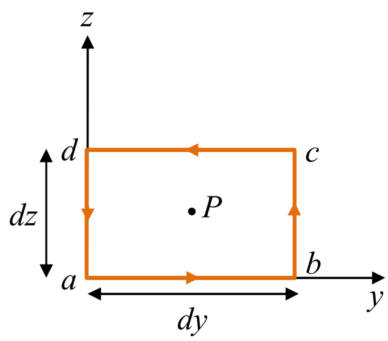
Curl of a Vector

On side ab, $d\vec{l} = dy\,\hat{a}_y$ and $z = z_0 - dz/2$

$$\oint_{ab} \vec{A}\cdot d\vec{l} = dy\left[A_y(x_0,y_0,z_0) - \frac{dz}{2}\frac{\partial A_y}{\partial z}\bigg|_P\right]$$

On side bc, $d\vec{l} = dz\,\hat{a}_z$ and $y = y_0 + dy/2$

$$\oint_{bc} \vec{A}\cdot d\vec{l} = dz\left[A_z(x_0,y_0,z_0) + \frac{dy}{2}\frac{\partial A_z}{\partial y}\bigg|_P\right]$$



Similarly,

$$\oint_{cd} \vec{A}\cdot d\vec{l} = -dy\left[A_y(x_0,y_0,z_0) + \frac{dz}{2}\frac{\partial A_y}{\partial z}\bigg|_P\right]$$

$$\oint_{da} \vec{A}\cdot d\vec{l} = -dz\left[A_z(x_0,y_0,z_0) - \frac{dy}{2}\frac{\partial A_z}{\partial y}\bigg|_P\right]$$

$$\lim_{\Delta S\to 0}\oint_L \frac{\vec{A}\cdot d\vec{l}}{\Delta S} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

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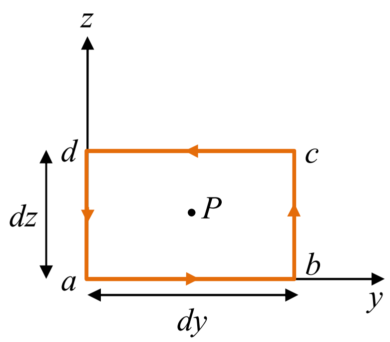
Curl of a Vector

$$\lim_{\Delta S \to 0} \oint_L \frac{\vec{A} \cdot d\vec{l}}{\Delta S} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$(\text{curl})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$(\text{curl})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$(\text{curl})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

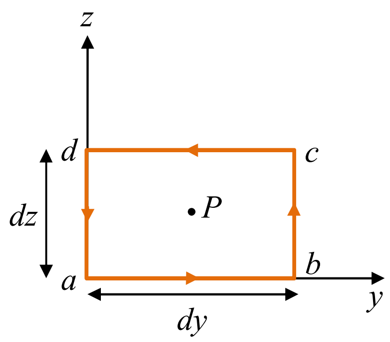


$$\nabla \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{a}_z$$

Curl of a Vector

In Cartesian coordinates

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$



Physical Significance

How much the vector curls around a point!

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Cylindrical Coordinates

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \vec{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \hat{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] \hat{a}_\phi + \frac{1}{\rho} \left[\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right] \hat{a}_z$$

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Spherical Coordinates

$$\nabla\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{a}_r & r\,\hat{a}_\theta & \hat{a}_\phi\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

$$\nabla\times\vec{A}=\frac{1}{r^2\sin\theta}\left[\frac{\partial\left(A_\phi\sin\theta\right)}{\partial\theta}-\frac{\partial A_\theta}{\partial\phi}\right]\hat{a}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial\left(rA_\phi\right)}{\partial r}\right]\hat{a}_\theta$$
$$+\frac{1}{r}\left[\frac{\partial\left(rA_\theta\right)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right]\hat{a}_\phi$$

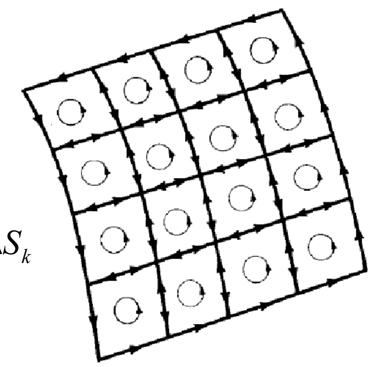
Stokes Theorem

The circulation of a vector field $\vec{A}$ around a (closed) path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S bounded by L provided that $\vec{A}$ and $\nabla\times\vec{A}$ are continuous on S.

$$\oint_L \vec{A}\cdot d\vec{l}=\int_S\left(\nabla\times\vec{A}\right)\cdot d\vec{S}$$

Proof:

$$\int_L \vec{A}\cdot d\vec{l}=\sum_k\oint_{L_k}\vec{A}\cdot d\vec{l}=\sum_k\frac{\oint_{L_k}\vec{A}\cdot d\vec{l}}{\Delta S_k}\Delta S_k$$
$$=\int_S\left(\nabla\times\vec{A}\right)\cdot d\vec{S}$$



Laplacian of a Scalar

The Laplacian of a scalar field V is the divergence of the gradient of V.

$$\text{Laplacian } V = \nabla \cdot \nabla V = \nabla^2 V$$

$$= \left[\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \right] \cdot \left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \right]$$

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

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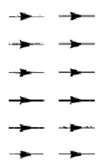
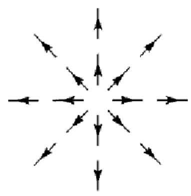
Divergence and Curl of Vector Fields



$\nabla \cdot \vec{A} = ?, \nabla \times \vec{A} = ?$



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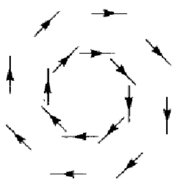

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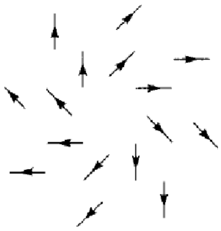

$\nabla \cdot \vec{A} = ?, \nabla \times \vec{A} = ?$

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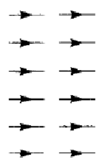
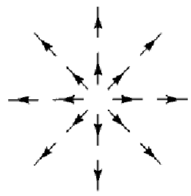
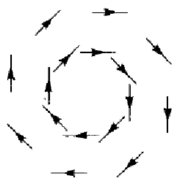
Divergence and Curl of Vector Fields



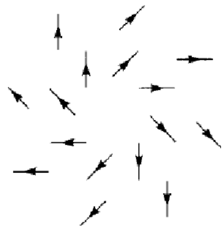
$\nabla \cdot \vec{A} = 0, \nabla \times \vec{A} = 0$



$\nabla \cdot \vec{A} \neq 0, \nabla \times \vec{A} = 0$



$\nabla \cdot \vec{A} = 0, \nabla \times \vec{A} \neq 0$



$\nabla \cdot \vec{A} \neq 0, \nabla \times \vec{A} \neq 0$