VECTOR CALCULUS

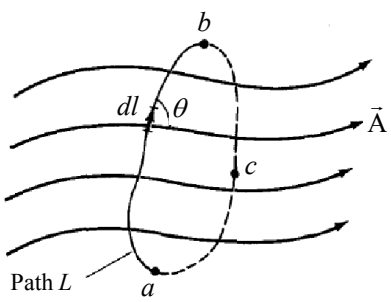
1

Line Integral

$$\int_L \vec{A} \cdot d\vec{l} = \int_a^b |\vec{A}| \cos\theta \, dl$$

If path of integration is closed

$$\oint_L \vec{A} \cdot d\vec{l}$$



2

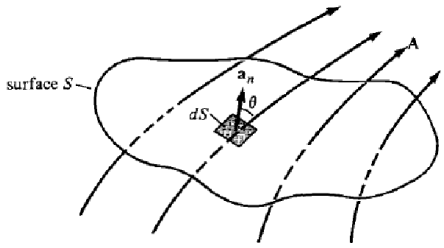
Surface Integral

$$\psi = \int_S \vec{A} \cdot d\vec{S} = \int_S \vec{A} \cdot \hat{a}_n \, dS = \int_S |\vec{A}| \cos\theta \, dS$$

$\hat{a}_n$ is the unit vector normal to S

For a closed surface

$$\psi = \oint_S \vec{A} \cdot d\vec{S}$$



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Volume Integral

$$Q = \int_v \rho_v dv$$

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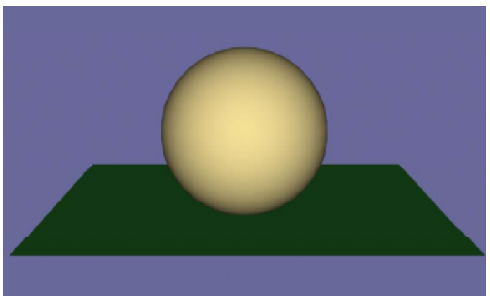
Closed Path and Closed Surface



A closed path defines a surface.

A closed surface defines a volume.

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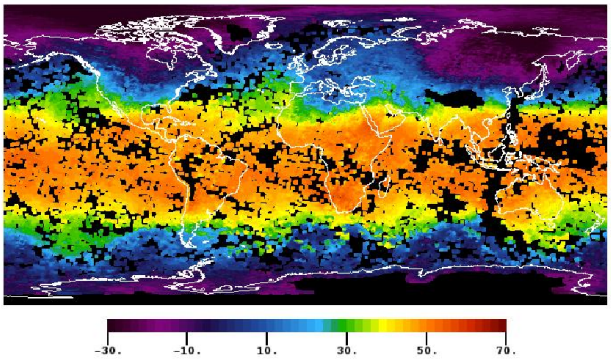
Gradient of a Scalar Field

Global night time air temperature



- ***What is the space rate of change of Temperature?***

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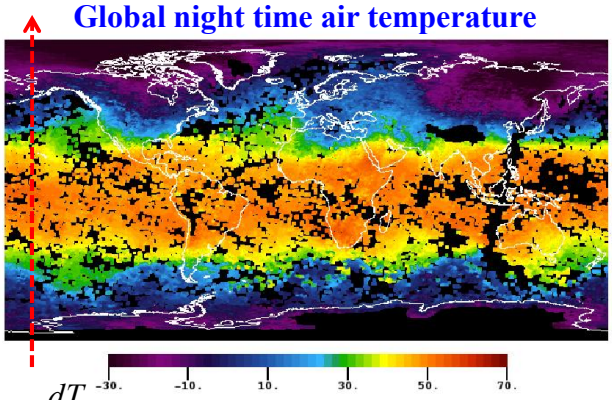
Gradient of a Scalar Field

Global night time air temperature



1D: $\dfrac{dT}{dz}$

- *If T is also a function of x and y! 3D.*
- *Space rate of change along a specified direction?*

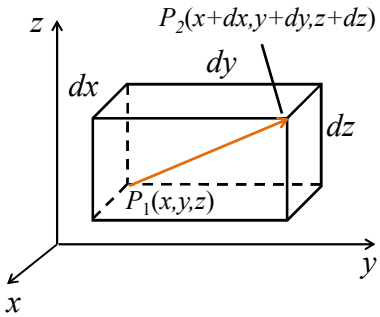
3D

$$d\vec{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$$

$$dT = \hat{a}_x \frac{\partial T}{\partial x} \cdot d\vec{l} + \hat{a}_y \frac{\partial T}{\partial y} \cdot d\vec{l} + \hat{a}_z \frac{\partial T}{\partial z} \cdot d\vec{l}$$

$$= \left(\hat{a}_x \frac{\partial T}{\partial x} + \hat{a}_y \frac{\partial T}{\partial y} + \hat{a}_z \frac{\partial T}{\partial z} \right) \cdot d\vec{l}$$

$$\nabla T = \text{grad}\, T = \hat{a}_x \frac{\partial T}{\partial x} + \hat{a}_y \frac{\partial T}{\partial y} + \hat{a}_z \frac{\partial T}{\partial z}$$



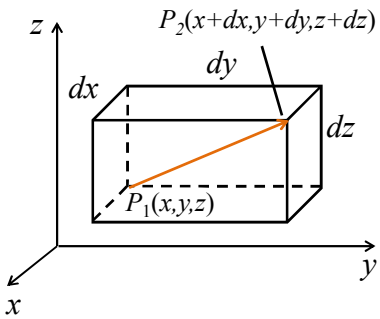
Del Operator

$$\nabla = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}$$

Del or gradient operator

$$dT = \nabla T \cdot d\vec{l}$$

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\vec{l}$$

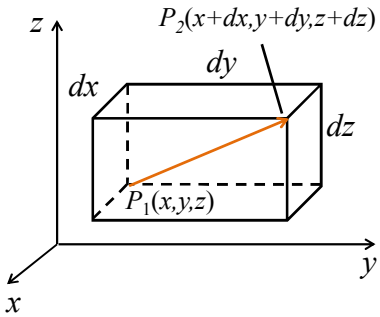


Directional Derivative

The projection of ∇T in the direction of a unit vector

$$dT = \nabla T \cdot \hat{a}_l dl$$

$$\frac{dT}{dl} = \nabla T \cdot \hat{a}_l$$



Example

Find the directional derivative of $T = x^2 + y^2 z$ along the direction $\hat{a}_x 2 + \hat{a}_y 3 - \hat{a}_z 2$ and evaluate it at (1, –1, 2).

$$\nabla T = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \right) \left(x^2 + y^2 z \right)$$

$$= \hat{a}_x 2x + \hat{a}_y 2yz + \hat{a}_z y^2$$

$$\vec{l} = \hat{a}_x 2 + \hat{a}_y 3 - \hat{a}_z 2$$

$$\hat{a}_l = \vec{l} / \left| \vec{l} \right| = \frac{\hat{a}_x 2 + \hat{a}_y 3 - \hat{a}_z 2}{\sqrt{2^2 + 3^2 + 2^2}}$$

$$= \frac{\hat{a}_x 2 + \hat{a}_y 3 - \hat{a}_z 2}{\sqrt{17}}$$

Example

Find the directional derivative of $T = x^2 + y^2 z$ along the direction $\hat{a}_x 2 + \hat{a}_y 3 - \hat{a}_z 2$ and evaluate it at (1, –1, 2).

$$\frac{dT}{dl} = \nabla T \cdot \hat{a}_l$$

$$= \left(\hat{a}_x 2x + \hat{a}_y 2yz + \hat{a}_z y^2 \right) \cdot \left(\frac{\hat{a}_x 2 + \hat{a}_y 3 - \hat{a}_z 2}{\sqrt{17}} \right)$$

$$\left. \frac{dT}{dl} \right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}$$

Cylindrical Coordinates

$$\rho = \sqrt{x^2 + y^2}, \quad \tan\phi = \frac{y}{x}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\phi$$

$$\frac{\partial \phi}{\partial x} = -\frac{1}{\rho}\sin\phi$$

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial y}$$

$$\frac{\partial \rho}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin\phi$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{\rho}\cos\phi$$

$$\frac{\partial z}{\partial y} = 0$$

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Cylindrical Coordinates

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}$$

$$\hat{a}_x = \cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi$$

$$\hat{a}_y = \sin\phi\,\hat{a}_\rho + \cos\phi\,\hat{a}_\phi$$

$$\hat{a}_z = \hat{a}_z$$

$$\boxed{\nabla = \hat{a}_\rho\frac{\partial}{\partial \rho} + \hat{a}_\phi\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{a}_z\frac{\partial}{\partial z}}$$

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Spherical Coordinates

$$r=\sqrt{x^2+y^2+z^2},\quad \tan\theta=\frac{\sqrt{x^2+y^2}}{z},\quad \tan\phi=\frac{y}{x}$$

$$\frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y}=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\boxed{\nabla=\hat{\mathrm{a}}_r\frac{\partial}{\partial r}+\hat{\mathrm{a}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\mathrm{a}}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}}$$

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Properties of Gradient Operator

- For any two scalar functions U and V

$$\nabla(U+V)=\nabla U+\nabla V$$

$$\nabla(UV)=U\nabla V+V\nabla U$$

$$\nabla V^n=nV^{n-1}\nabla V$$

- ***The gradient of a vector is meaningless under the rules of vector calculus.***

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Divergence of a Vector

The divergence of A at a given point P is the outward flux per unit volume as the volume shrinks about P.

$$\operatorname{div}\vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{S}}{\Delta v}$$

where Δv is the volume enclosed by the closed surface S in which P is located.

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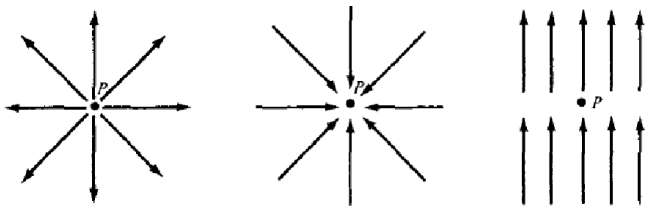
Divergence of a Vector

Positive divergence Negative divergence Zero divergence



The divergence of a vector field can be viewed as simply the limit of the field's source strength per unit volume (or source density); it is positive at a source point in the field, and negative at a sink point, or zero where there is neither sink nor source.

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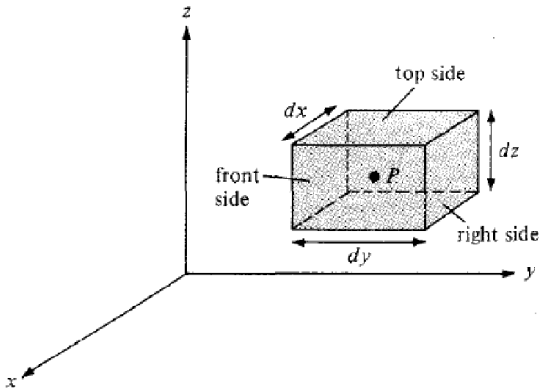
Divergence of a Vector

$\nabla \cdot \vec{A}$ in Cartesian coordinates at a point $P(x_0, y_0, z_0)$

$$\oint_S \vec{A} \cdot d\vec{S} = \left(\int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}} \right) \vec{A} \cdot d\vec{S}$$



Divergence of a Vector

3-D Taylor series expansion of A_x about P

$$A_x(x,y,z) = A_x(x_0,y_0,z_0) + (x - x_0)\left.\frac{\partial A_x}{\partial x}\right|_P + (y - y_0)\left.\frac{\partial A_x}{\partial y}\right|_P + (z - z_0)\left.\frac{\partial A_x}{\partial z}\right|_P + \text{higher-order terms}$$

Divergence of a vector in Cartesian coordinates

$$\boxed{\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Divergence Theorem

Total outward flux of a vector field $\vec{A}$ through the closed surface S is the same as the volume integral of the divergence of $\vec{A}$.

$$\oint_S \vec{A} \cdot d\vec{S} = \int_v \nabla \cdot \vec{A}\, dv$$

Proof:

$$\oint_S \vec{A} \cdot d\vec{S} = \sum_k \oint_{S_k} \vec{A} \cdot d\vec{S} = \sum_k \frac{\oint_{S_k} \vec{A} \cdot d\vec{S}}{\Delta v_k} \Delta v_k = \int_v \nabla \cdot \vec{A}\, dv$$

Volume integrals are easier to evaluate than surface integrals!

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