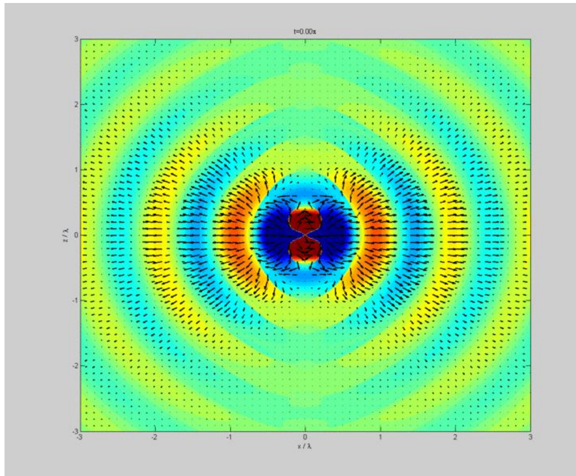
POYNTING VECTOR AND ENERGY FLOW

Poynting Vector

$$\vec{P} = \vec{E} \times \vec{H}$$

The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.

Poynting Vector and Energy Flow



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Time Averaged Poynting Vector

$$
\begin{aligned}
\vec{\mathrm{P}} &= \vec{\mathrm{E}} \times \vec{\mathrm{H}} \\
&= \mathrm{Re}\left(\tilde{E}\right) \times \mathrm{Re}\left(\tilde{H}\right) \\
&= \mathrm{Re}\left(E_s e^{j\omega t}\right) \times \mathrm{Re}\left(H_s e^{j\omega t}\right) \\
&= \frac{1}{2}\left(E_s e^{j\omega t} + E_s^* e^{-j\omega t}\right) \times \frac{1}{2}\left(H_s e^{j\omega t} + H_s^* e^{-j\omega t}\right) \\
&= \frac{1}{4}\left(E_s \times H_s^* + E_s^* \times H_s + E_s \times H_s e^{2j\omega t} + E_s^* \times H_s^* e^{-2j\omega t}\right) \\
&= \frac{1}{4}\left(E_s \times H_s^* + \left(E_s \times H_s^*\right)^* + E_s \times H_s e^{2j\omega t} + \left(E_s \times H_s e^{2j\omega t}\right)^*\right) \\
&= \frac{1}{2}\mathrm{Re}\left(E_s \times H_s^*\right) + \frac{1}{2}\mathrm{Re}\left(E_s \times H_s e^{2j\omega t}\right)
\end{aligned}
$$

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Time Averaged Poynting Vector

The average over time is given as

$$\langle P \rangle = \frac{1}{T}\int_0^T P(t)dt$$

$$= \frac{1}{T}\int_0^T \left[\frac{1}{2}\mathrm{Re}\left(E_s \times H_s^*\right) + \frac{1}{2}\mathrm{Re}\left(E_s \times H_s e^{2j\omega t}\right)\right]dt$$

The second term is a sinusoid curve

$$\mathrm{Re}\left(e^{2j\omega t}\right) = \cos 2\omega t$$

And its average is zero, giving

$$\langle P \rangle = \frac{1}{2}\mathrm{Re}\left(E_s \times H_s^*\right) = \frac{1}{2}\mathrm{Re}\left[\left(E_s e^{j\omega t}\right) \times \left(H_s^* e^{-j\omega t}\right)\right]$$

$$= \frac{1}{2}\mathrm{Re}\left(\tilde{E} \times \tilde{H}^*\right)$$

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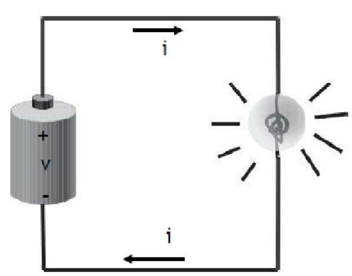
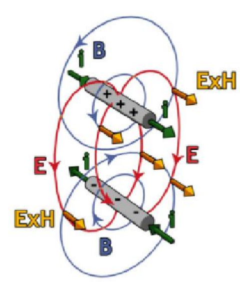
Power Flow in a Simple Circuit





Due to current flow through the conductor →

$$E_z = \frac{V}{d} = \frac{IR}{d} \qquad H_\phi = \frac{I}{2\pi r}$$

$$P = E \times H = -E_z H_\phi \,\hat{a}_r$$

$$= -\frac{IR}{d}\frac{I}{2\pi r}\hat{a}_r = -\frac{I^2 R}{2\pi r d}\hat{a}_r$$

Power flows inward to conductor.

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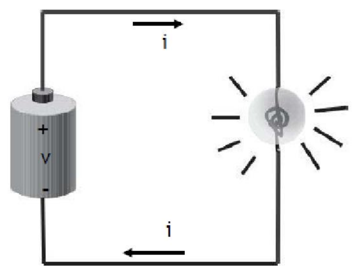
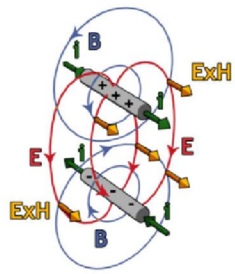
Power Flow in a Simple Circuit





$$P = E \times H = -E_z H_\phi \hat{a}_r$$

$$= -\frac{IR}{d}\frac{I}{2\pi r}\hat{a}_r = -\frac{I^2 R}{2\pi r d}\hat{a}_r$$

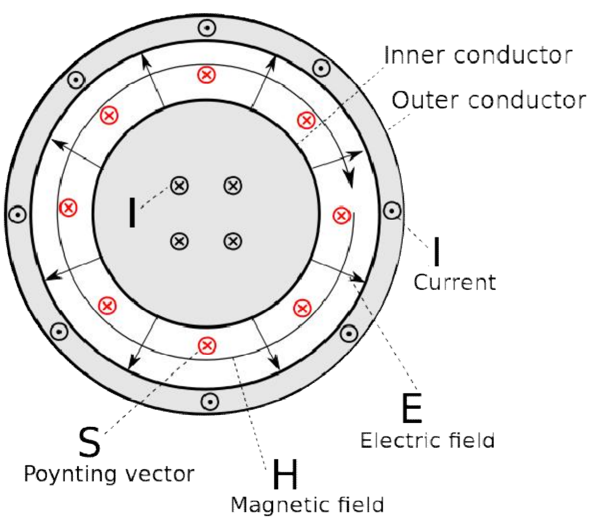
Total power entering the conductor:

$$P_{\text{tot}} = \oiint P \cdot dS \approx -\frac{I^2 R}{2\pi r d} 2\pi r d$$

$$= I^2 R$$

Agreement with circuit theory.

Coaxial Cable



ELECTROMAGNETIC WAVE PROPAGATION

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Media

Solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space $(\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0)$
2. Lossless dielectric $(\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0, \text{or } \sigma \ll \omega\varepsilon)$
3. Lossy dielectrics $(\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0)$
4. Good conductors $(\sigma \simeq \infty, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0, \text{or } \sigma \gg \omega\varepsilon)$

- Lossy dielectric media is the most general one.
- Others are special cases.

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EM Waves in Lossy Dielectrics

Lossy dielectric → EM wave loses power as it propagates.

Consider a charge free ($\rho_v = 0$) lossy dielectric.

Maxwell's equations:

$$\nabla \cdot \vec{D}_S = \rho_v$$
$$\nabla \cdot \vec{B}_S = 0$$
$$\nabla \times \vec{E}_S = -j\omega \vec{B}_S$$
$$\nabla \times \vec{H}_S = \vec{J}_S + j\omega \vec{D}_S$$

→

$$\nabla \cdot \vec{E}_s = 0$$
$$\nabla \cdot \vec{H}_s = 0$$
$$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$$
$$\nabla \times \vec{H}_s = (\sigma + j\omega\varepsilon)\vec{E}_s$$

Helmholtz Equations

Taking curl

$$\nabla \times \nabla \times \vec{E}_s = -j\omega\mu \nabla \times \vec{H}_s$$
$$\nabla(\nabla \cdot \vec{E}_s) - \nabla^2 \vec{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}_s$$
$$\nabla^2 \vec{E}_s - \gamma^2 \vec{E}_s = 0$$

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

γ: Propagation constant

$$\nabla \cdot \vec{E}_s = 0$$
$$\nabla \cdot \vec{H}_s = 0$$
$$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$$
$$\nabla \times \vec{H}_s = (\sigma + j\omega\varepsilon)\vec{E}_s$$

Using a similar procedure: $\nabla^2 \vec{H}_s - \gamma^2 \vec{H}_s = 0$

Propagation Constant

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = -\omega^2\mu\varepsilon + j\omega\mu\sigma$$

$$\gamma = \alpha + j\beta$$

$$\alpha = ? \quad \beta = ?$$

$$-\mathrm{Re}\left(\gamma^2\right) = \beta^2 - \alpha^2 = \omega^2\mu\varepsilon$$

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]} \qquad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

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Propagating Waves

Assume that the wave propagates in *z* and has only *x*-component

$$\vec{\mathrm{E}}_s = E_{xs}(z)\hat{\mathrm{a}}_x \Rightarrow \left(\nabla^2 - \gamma^2\right)E_{xs}(z) = 0$$

$$\frac{\partial^2 E_{xs}(z)}{\partial x^2} + \frac{\partial^2 E_{xs}(z)}{\partial y^2} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

$$\left[\frac{d^2}{dz^2} - \gamma^2\right]E_{xs}(z) = 0$$

$$\Downarrow$$

$$E_{xs}(z) = E_0 e^{-\gamma z} + E_0' e^{\gamma z}$$

For the field to be finite at infinity, $E_0' = 0$.

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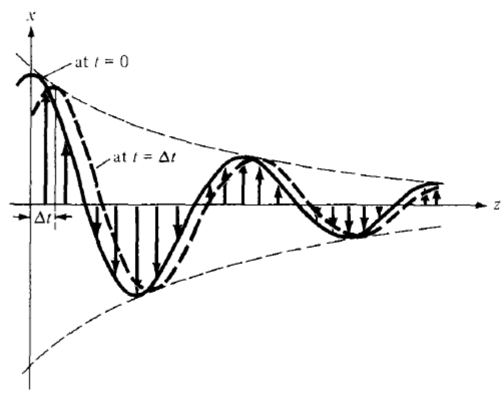
Propagating Waves

$$\vec{E}(z,t) = \mathrm{Re}\left[E_{xs}(z)e^{j\omega t}\,\hat{a}_x\right] = \mathrm{Re}\left(E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{a}_x\right)$$

$$\vec{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\,\hat{a}_x$$



Propagating Magnetic Field

$$\vec{E}_S(z) = E_0 e^{-\gamma z}\,\hat{a}_x \qquad \vec{H}(z,t) = ?$$

$$\nabla \times \vec{E}_S = -j\omega\mu \vec{H}_S \Rightarrow \vec{H}_S = \frac{j}{\omega\mu}\nabla \times \vec{E}_S = \frac{j}{\omega\mu}\left(-\gamma E_0 e^{-\gamma z}\,\hat{a}_y\right)$$

$$\vec{H}_S(z) = H_0 e^{-\gamma z}\,\hat{a}_y$$

$$H_0(z) = \frac{E_0}{\eta} = -j\frac{\gamma}{\omega\mu}E_0 \Rightarrow \eta = j\frac{\omega\mu}{\gamma}$$

η : intrinsic impedance

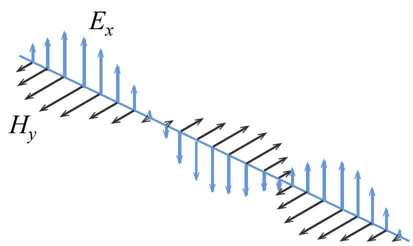
$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta = |\eta|\,e^{j\theta_\eta}$$

Intrinsic Impedance

$$\eta = |\eta| \angle\theta_\eta = |\eta| e^{j\theta_\eta}$$

$$\vec{H}(z,t) = \text{Re}\left(H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\, \hat{a}_y \right)$$

$$= \text{Re}\left[\frac{E_0}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)}\, \hat{a}_y \right]$$



At any instant E and H are out of phase by θ_η.

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