POYNTING VECTOR AND ENERGY FLOW

1

Time-Harmonic Fields

- Let's assume that the fields are time harmonic.
- A time-harmonic field is one that varies periodically or sinusoidally with time.

$$\tilde{A} = \vec{A}_0 e^{j(\omega t - \beta x)}$$

$$\tilde{A} = \vec{A}_s e^{j\omega t}$$

$$\vec{A} = \mathrm{Re}\left(\vec{A}_s e^{j\omega t}\right)$$

2

Time-Harmonic Fields

$$\frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t}\mathrm{Re}\left(\vec{A}_S\, e^{j\omega t}\right) = \mathrm{Re}\left(j\omega \vec{A}_S\, e^{j\omega t}\right)$$

$$\frac{\partial \vec{A}}{\partial t} \rightarrow j\omega \vec{A}_S$$

$$\int \vec{A}\,\partial t \rightarrow \frac{\vec{A}_S}{j\omega}$$

Time-Harmonic Maxwell's Equations

Differential Form	**Integral Form**
$\nabla \cdot \vec{D}_S = \rho_v$	$\oint_S \vec{D}_S \cdot d\vec{S} = \int_v \rho_{vS}\, dv$
$\nabla \cdot \vec{B}_S = 0$	$\oint_S \vec{B}_S \cdot d\vec{S} = 0$
$\nabla \times \vec{E}_S = -j\omega \vec{B}_S$	$\oint_L \vec{E}_S \cdot d\vec{l} = -j\omega \int_S \vec{B}_S \cdot d\vec{S}$
$\nabla \times \vec{H}_S = \vec{J}_S + j\omega \vec{D}_S$	$\oint_L \vec{H}_S \cdot d\vec{l} = \int_S \left(\vec{J}_S + j\omega \vec{D}_S\right) \cdot d\vec{S}$

Energy Flow

Rate of energy transportation is obtained from Maxwell's equations:

$$\nabla\times\vec{H}=\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$\vec{E}\cdot$ ⇓

$$\vec{E}\cdot\left(\nabla\times\vec{H}\right)=\sigma E^2+\vec{E}\cdot\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$$

$\vec{H}\cdot$ ⇓

$$\vec{H}\cdot(\nabla\times\vec{E})=\vec{H}\cdot\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)=-\frac{\mu}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H})$$

5

Energy Flow

For vector fields $\vec{A}$ and $\vec{B}$, $\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$

Let $\vec{A}=\vec{H},\ \vec{B}=\vec{E}$ $\Rightarrow$ $\nabla\cdot(\vec{H}\times\vec{E})=\vec{E}\cdot(\nabla\times\vec{H})-\vec{H}\cdot(\nabla\times\vec{E})$

$$\nabla\cdot(\vec{H}\times\vec{E})=\sigma E^2+\vec{E}\cdot\varepsilon\frac{\partial\vec{E}}{\partial t}+\frac{\mu}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H})$$

$$\nabla\cdot(\vec{H}\times\vec{E})=\sigma E^2+\frac{\varepsilon}{2}\frac{\partial E^2}{\partial t}+\frac{\mu}{2}\frac{\partial H^2}{\partial t}$$

6

Energy Flow

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\sigma E^2 - \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t} - \frac{\mu}{2}\frac{\partial H^2}{\partial t}$$

Taking volume integral

$$\int_v \nabla\cdot\left(\vec{E}\times\vec{H}\right) dv = -\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2\, dv$$

Applying divergence theorem

$$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S} = -\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2\, dv$$

Energy Flow

$$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S} = -\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2\, dv$$

$-\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv$: Rate of decrease in energy stored in electric and magnetic field

$-\int_v \sigma E^2\, dv$: Ohmic power dissipated

$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S}$: Total power leaving the volume

Poynting Vector

$$\vec{P} = \vec{E} \times \vec{H}$$

The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.