COORDINATE SYSTEMS AND TRANSFORMATIONS

1

Coordinate Systems

- Physical quantities must be defined in space suitably.
- Classifications:
  - Orthogonal → coordinates are mutually perpendicular.
    - a) Cartesian
    - b) Circular cylindrical
    - c) Spherical
  - Nonorthogonal → little or no practical use.

Save time and work by choosing a coordinate system that best fits a given problem!

2

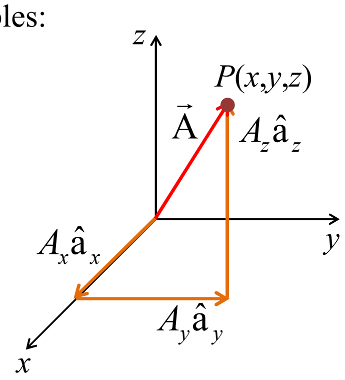
Cartesian Coordinates (x,y,z)

- A point *P* can be represented as (*x*,*y*,*z*).
- The ranges of the coordinate variables:

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

- $\vec{A} = (A_x, A_y, A_z)$, or

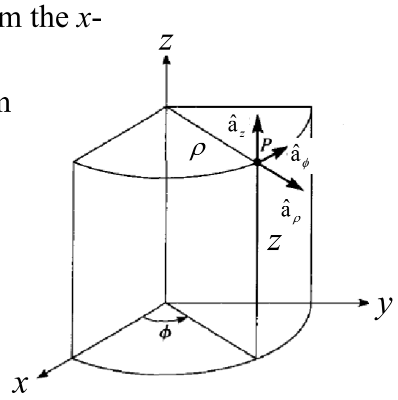
$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$



Circular Cylindrical Coordinates (ρ,ϕ,z)

- Point *P* can be represented as (ρ,ϕ,z)
  - ρ : radius of the cylinder through *P*
  - ϕ : azimuthal angle, measured from the *x*-axis in the *xy*-plane
  - *z* : same as in the Cartesian system
- The ranges of the variables:

$$0 \le \rho < \infty$$
$$0 \le \phi < 2\pi$$
$$-\infty < z < \infty$$

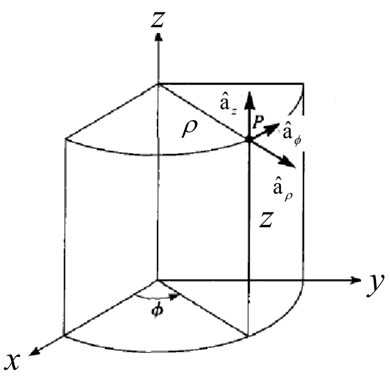


Circular Cylindrical Coordinates (ρ,ϕ,z)

- $\vec{A} = (A_\rho, A_\phi, A_z)$, or

  $\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$

  $|\vec{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$



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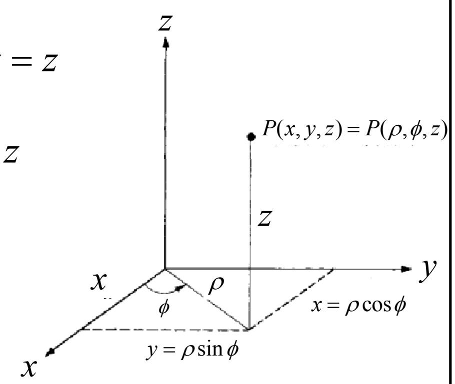
$(x,y,z) \leftrightarrow (\rho,\phi,z)$

$$\rho = \sqrt{x^2 + y^2},\ \phi = \tan^{-1}\frac{y}{x},\ z = z$$

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$



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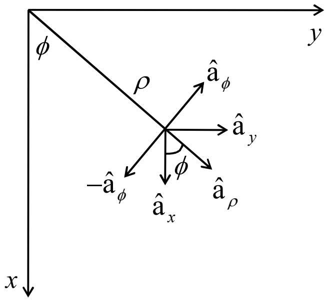
$(\hat{a}_x, \hat{a}_y, \hat{a}_z) \leftrightarrow (\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z)$

$$\hat{a}_\rho \cdot \hat{a}_x = \cos\phi, \quad \hat{a}_\rho \cdot \hat{a}_y = \sin\phi$$
$$\hat{a}_\phi \cdot \hat{a}_x = -\sin\phi, \quad \hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$



Let

$$\hat{a}_\rho = \hat{a}_x\alpha + \hat{a}_y\beta$$
$$\hat{a}_\rho \cdot \hat{a}_x = \hat{a}_x \cdot \hat{a}_x\alpha + \hat{a}_y \cdot \hat{a}_x\beta = \alpha$$
$$\hat{a}_\rho \cdot \hat{a}_y = \hat{a}_x \cdot \hat{a}_y\alpha + \hat{a}_y \cdot \hat{a}_y\beta = \beta$$

$$\hat{a}_\rho = \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y$$
$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$$
$$\hat{a}_z = \hat{a}_z$$

$$\hat{a}_x = \cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi$$
$$\hat{a}_y = \sin\phi\,\hat{a}_\rho + \cos\phi\,\hat{a}_\phi$$
$$\hat{a}_z = \hat{a}_z$$

$(A_x, A_y, A_z) \leftrightarrow (A_\rho, A_\phi, A_z)$

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z = A_\rho\hat{a}_\rho + A_\phi\hat{a}_\phi + A_z\hat{a}_z$$

Substitute Cartesian unit vectors by cylindrical unit vectors

$$\vec{A} = \left(A_x\cos\phi + A_y\sin\phi\right)\hat{a}_\rho + \left(-A_x\sin\phi + A_y\cos\phi\right)\hat{a}_\phi + A_z\hat{a}_z$$

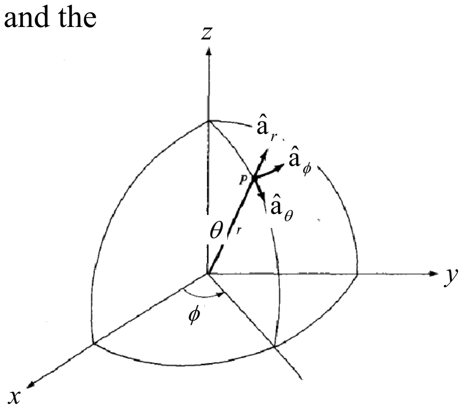
$$A_\rho = A_x\cos\phi + A_y\sin\phi$$
$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$
$$A_z = A_z$$

$$A_x = A_\rho\cos\phi - A_\phi\sin\phi$$
$$A_y = A_\rho\sin\phi + A_\phi\cos\phi$$
$$A_z = A_z$$

Spherical Coordinates (r,θ,φ)

- A point P can be represented as (r, θ, ϕ):
  - r : distance from the origin to point P
  - θ: angle between the z-axis and the position vector of P
  - ϕ: angle from the x-axis
- The ranges of the variables

$$0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \phi < 2\pi$$



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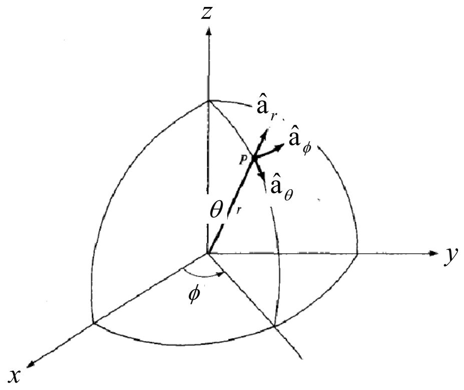
Spherical Coordinates (r,θ,φ)

$$\vec{A} = (A_r, A_\theta, A_\phi), \text{or}$$
$$\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$$
$$|\vec{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$$



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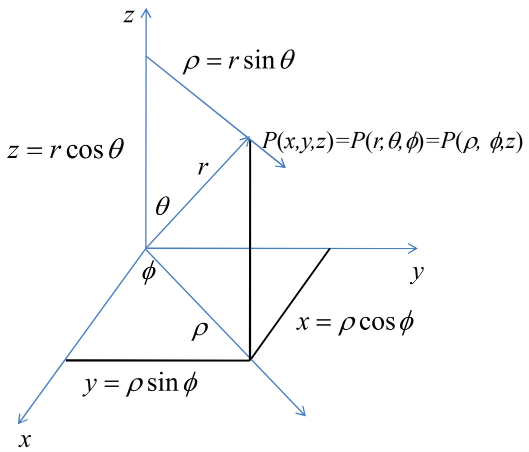
$(x,y,z) \leftrightarrow (r,\theta,\phi)$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$



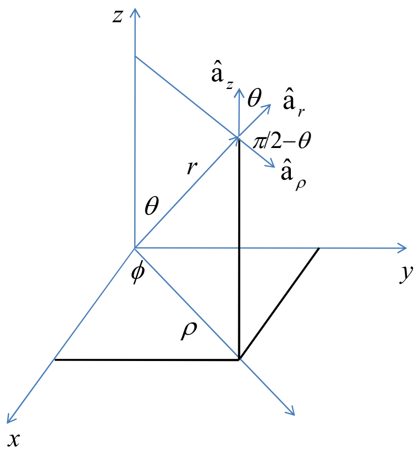
$(\hat{a}_x, \hat{a}_y, \hat{a}_z) \leftrightarrow (\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi)$

$\hat{a}_r$ lies in the $\hat{a}_\rho$ and $\hat{a}_z$ plane

$$\hat{a}_r = \hat{a}_\rho \alpha + \hat{a}_z \beta$$

$$\hat{a}_r \cdot \hat{a}_\rho = \alpha$$
$$\hat{a}_r \cdot \hat{a}_z = \beta$$

$$\hat{a}_r \cdot \hat{a}_\rho = \sin\theta$$
$$\hat{a}_r \cdot \hat{a}_z = \cos\theta$$



$(\hat{a}_x, \hat{a}_y, \hat{a}_z) \leftrightarrow (\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z)$

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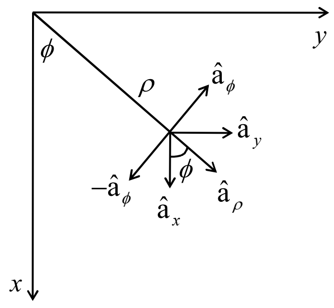
$$\hat{a}_\phi \cdot \hat{a}_x = -\sin\phi, \quad \hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$

Let

$$\hat{a}_\rho = \hat{a}_x\alpha + \hat{a}_y\beta$$
$$\hat{a}_\rho \cdot \hat{a}_x = \hat{a}_x \cdot \hat{a}_x\alpha + \hat{a}_y \cdot \hat{a}_x\beta = \alpha$$
$$\hat{a}_\rho \cdot \hat{a}_y = \hat{a}_x \cdot \hat{a}_y\alpha + \hat{a}_y \cdot \hat{a}_y\beta = \beta$$



$$\hat{a}_\rho = \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y$$
$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$$
$$\hat{a}_z = \hat{a}_z$$

$$\hat{a}_x = \cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi$$
$$\hat{a}_y = \sin\phi\,\hat{a}_\rho + \cos\phi\,\hat{a}_\phi$$
$$\hat{a}_z = \hat{a}_z$$

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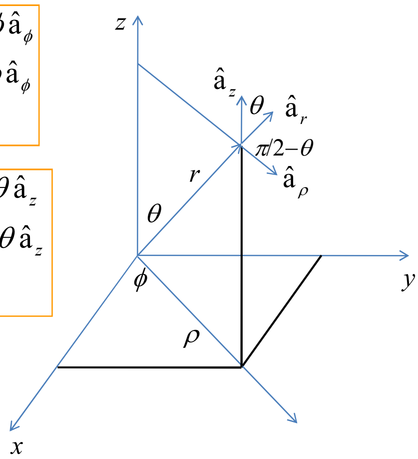
$(\hat{a}_x, \hat{a}_y, \hat{a}_z) \leftrightarrow (\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi)$

$$\hat{a}_x = \sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi$$
$$\hat{a}_y = \sin\theta\sin\phi\,\hat{a}_r + \cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi$$
$$\hat{a}_z = \cos\theta\,\hat{a}_r - \sin\theta\,\hat{a}_\theta$$

$$\hat{a}_r = \sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z$$
$$\hat{a}_\theta = \cos\theta\cos\phi\,\hat{a}_x + \cos\theta\sin\phi\,\hat{a}_y - \sin\theta\,\hat{a}_z$$
$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$$



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VECTOR CALCULUS

Differential Length, Area, and Volume

Cartesian Coordinates

1) Differential displacement

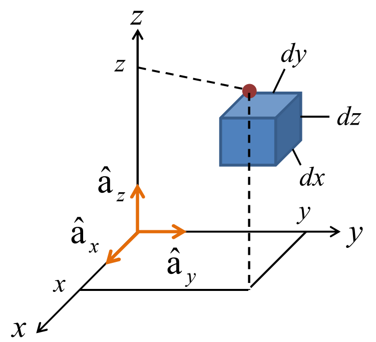
$$d\vec{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$

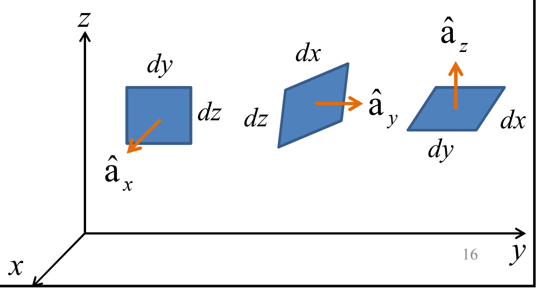
2) Differential normal area

$$d\vec{S} = dy\,dz\,\hat{a}_x$$
$$dx\,dz\,\hat{a}_y$$
$$dz\,dx\,\hat{a}_z$$

3) Differential volume

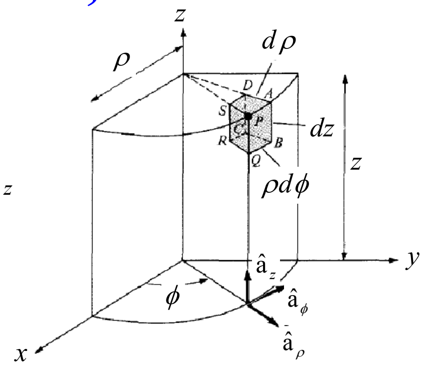
$$dv = dx\,dy\,dz$$





Differential Length, Area, and Volume

Cylindrical Coordinates

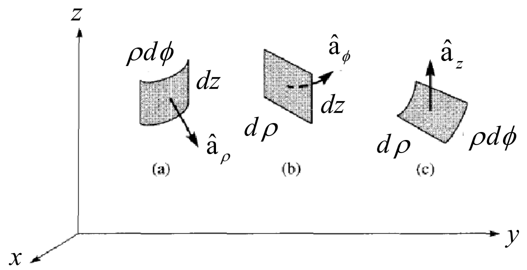


1) Differential displacement

$$d\vec{l} = d\rho\,\hat{a}_\rho + \rho\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$

2) Differential normal area

$$d\vec{S} = \rho\,d\phi\,dz\,\hat{a}_\rho$$
$$d\rho\,dz\,\hat{a}_\phi$$
$$\rho\,d\phi\,d\rho\,\hat{a}_z$$

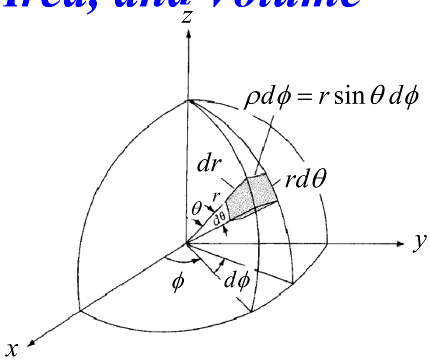


2) Differential volume

$$dv = \rho\,d\rho\,d\phi\,dz$$

Differential Length, Area, and Volume

Spherical Coordinates

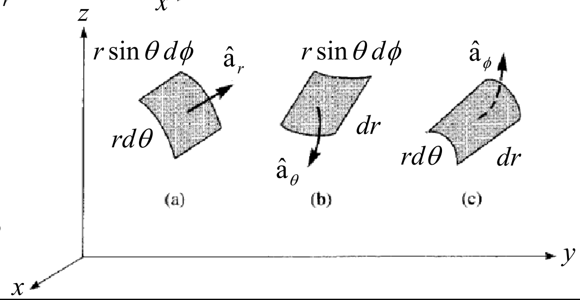


1) Differential displacement

$$d\vec{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$

2) Differential normal area

$$d\vec{S} = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r$$
$$r\sin\theta\,dr\,d\phi\,\hat{a}_\theta$$
$$r\,dr\,d\theta\,\hat{a}_\phi$$



2) Differential volume

$$dv = r^2\sin\theta\,dr\,d\theta\,d\phi$$