ELECTROMAGNETIC WAVES

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Electromagnetic Potentials

Electrostatics: $\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$

What happens when the fields are time-varying?

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) \Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Since the curl of the gradient of a scalar field is zero

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

$$\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}$$

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Time-Varying Potentials

Maxwell's equation → $\nabla \cdot \vec{D} = \rho_v \Rightarrow \nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon}$

Electromagnetic potential → $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$

Taking divergence → $\nabla \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{\rho_v}{\varepsilon}$

Maxwell's equation → $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \nabla \times \vec{B} = \mu\vec{J} + \varepsilon\mu\frac{\partial \vec{E}}{\partial t}$

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \mu\vec{J} + \varepsilon\mu\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= \mu\vec{J} - \mu\varepsilon\nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon\frac{\partial^2 \vec{A}}{\partial t^2}$$

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Time-Varying Potentials

$$\nabla \times \nabla \times \vec{A} = \mu\vec{J} - \mu\varepsilon\nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon\frac{\partial^2 \vec{A}}{\partial t^2}$$

Vector identity → $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$

$$\nabla^2\vec{A} - \nabla(\nabla \cdot \vec{A}) = -\mu\vec{J} + \mu\varepsilon\nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon\frac{\partial^2 \vec{A}}{\partial t^2}$$

Lorentz condition for potentials →

$$\nabla \cdot \vec{A} = -\mu\varepsilon\frac{\partial V}{\partial t}$$

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Wave Equations

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J}$$

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Maxwell's Equations

Free space with charges

	Differential form	Integral form
E-Gauss:	$\nabla \cdot \epsilon_o \vec{E} = \rho$	$\oiint_S \epsilon_o \vec{E} \cdot dS = \iiint_V \rho dV$
Faraday:	$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\mu_o \vec{H}$	$\oint_C \vec{E} \cdot d\vec{C} = \iint_S -\frac{\partial}{\partial t}\mu_o \vec{H} \cdot d\vec{S}$
H-Gauss:	$\nabla \cdot \mu_o \vec{H} = 0$	$\oiint_S \mu_o \vec{H} \cdot d\vec{S} = 0$
Ampere:	$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\epsilon_o \vec{E}$	$\oint_C \vec{H} \cdot d\vec{C} = \iint_S (\vec{J} + \frac{\partial}{\partial t}\mu_o \vec{H}) \cdot d\vec{S}$

In statics, both time derivatives are unimportant, Maxwell's Equations split into decoupled electrostatic and magnetostatic equations.
In Electro-quasistatic (EQS) and magneto-quasitatic systems (MQS), one (but not both) time derivative becomes important.

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Coupling of Electric and Magnetic Fields

Maxwell's Equations couple the E and H fields:

$$\oint_C \overline{E}\cdot d\bar{l} = -\frac{d}{dt}\left(\int_S \overline{B}\cdot d\overline{A}\right)$$

$$\oint_C \overline{H}\cdot d\bar{l} = \int_S \overline{J}\cdot d\overline{A} + \frac{d}{dt}\int_S \epsilon E dA$$

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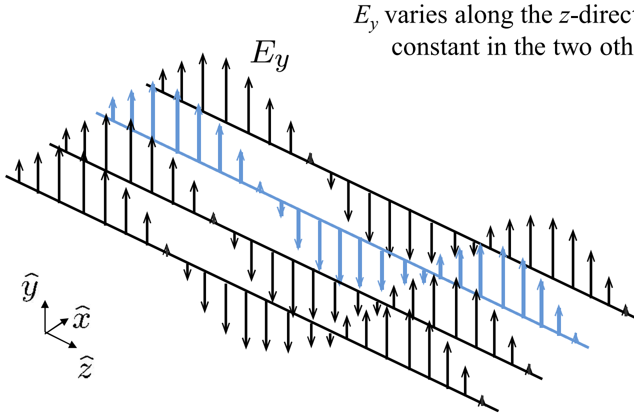
Uniform Electromagnetic Waves

E_y varies along the z-direction and E is constant in the two other directions



$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B}\cdot d\vec{A}\right)$$

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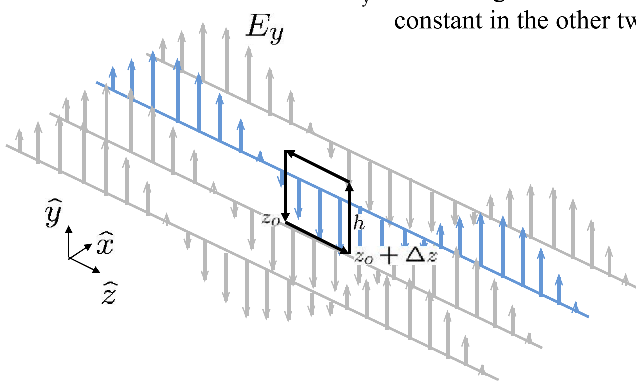
Uniform Electromagnetic Waves

E_y varies along the z-direction and E_y is constant in the other two directions



$$\oint_C \vec{E} \cdot d\vec{l} = \quad hE_y(z_o + \Delta z) - hE_y(z_o) \quad = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)$$

$$= h\Delta z \, \frac{E_y(z_o + \Delta z) - E_y(z_o)}{\Delta z} \quad = h\Delta z \frac{\partial E_y}{\partial z}$$

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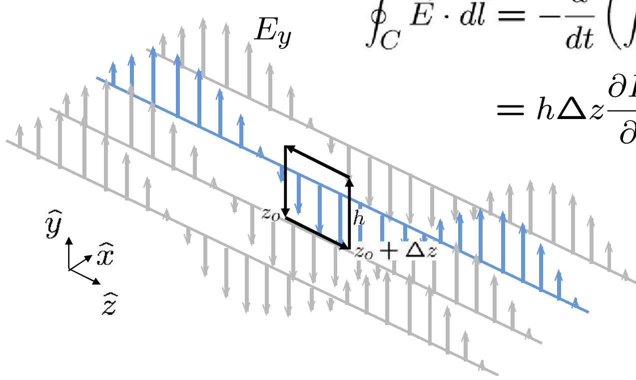
Electromagnetic Waves



$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)$$

$$= h\Delta z \frac{\partial E_y}{\partial z}$$

E_y-field cannot vary in z-direction without a time-varying B-field …

$$-\frac{\partial}{\partial t}\int \vec{B} d\vec{A} = -\frac{\partial}{\partial t}(-B_x h \Delta z) = h\Delta z \frac{\partial B_x}{\partial t}$$

…and waves must have both electric and magnetic components !

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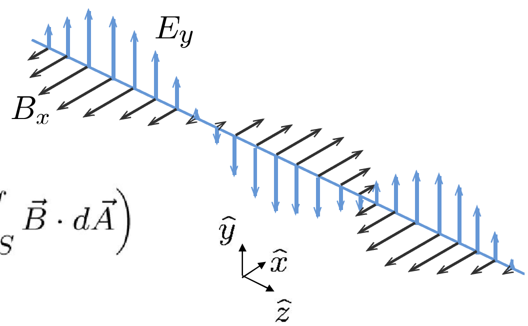
Uniform Electromagnetic Plane Waves

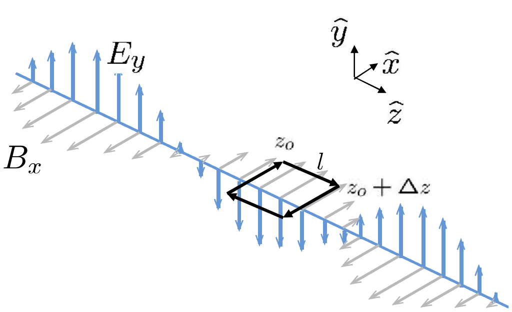


$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)$$

$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}$$

The **y-component of E** that varies across space is associated with the **x-component of B** that varies in time

Uniform Electromagnetic Plane Waves



$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{A} + \frac{d}{dt}\int_S \epsilon_o \vec{E}\, d\vec{A}$$

$$\frac{\partial B_x(z_o)}{\partial z} = \epsilon_o \mu_o \frac{\partial E_y}{\partial t}$$ Source free: $J = 0$

The Wave Equation

Time-varying E_y generates spatially varying B_z …

$$\frac{\partial^2 B_x(z_o)}{\partial z\,\partial t} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}$$

Time-varying B_z generates spatially varying E_y …

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{\partial^2 B_x(z_o)}{\partial t\,\partial z}$$

The temporal and spatial variations in E_y are coupled together to yield …

$$\boxed{\frac{\partial^2 E_y}{\partial z^2} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}}$$

… the Wave Equation.

Wave Equation via Differential Equations

Faraday:

$$\underbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}}_{\nabla \times \vec{E}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\hat{x}\frac{\partial E_y}{\partial z} = -\hat{x}\frac{\partial \mu H_x}{\partial t}$$

Ampere:

$$\underbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}}_{\nabla \times \vec{H}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = -\hat{y}\frac{\partial H_x}{\partial z} = -\hat{y}\frac{\partial \epsilon E_y}{\partial t}$$

Substitution yields the wave equation:

$$\frac{\partial^2 E_y}{\partial z^2} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}$$

Uniform Plane Wave Solutions

The 1-D wave equation

$$\frac{\partial^2 E_y}{\partial z^2} - \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2} = 0$$

- $E_y(z,t)$ is any function for which the second derivative in space equals its second derivative in time, times a constant. The solution is therefore any function with the same dependence on time as on space, e.g.

$$E_y = f_+(t - z/c) + f_-(t + z/c)$$

- The functions ***$f_+(z-ct)$*** and ***$f_-(z+ct)$*** represent uniform waves propagating in the *+z* and *-z* directions respectively.

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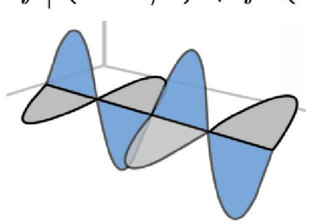
Speed of Light

- The ***velocity of propagation*** is determined solely by the dielectric permittivity and magnetic permeability:

$$c = \frac{1}{\sqrt{\epsilon_o \mu_o}}$$

- The functions ***f_+*** and ***f_-*** are determined by the source and the other boundary conditions.

$$E_y = f_+(t - z/c) + f_-(t + z/c)$$



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