DISPLACEMENT CURRENT

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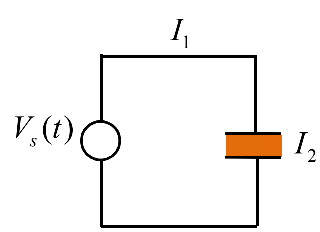
Current Through a Capacitor

How currents flow through a capacitor?



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Continuity Equation

Charge conservation:

$$I_{\text{out}} = \oint \vec{J} \cdot d\vec{S} = -\frac{dQ}{dt}$$

From divergence theorem:

$$\oint_S \vec{J} \cdot d\vec{S} = \int_v \nabla \cdot \vec{J} dv$$

$$-\frac{dQ}{dt} = -\frac{d}{dt}\int_v \rho_v dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

$$\int_v \nabla \cdot \vec{J} dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

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Displacement Current

Maxwell's equations for static EM → $\nabla \times \vec{H} = \vec{J}$

Taking divergence → $\nabla \cdot (\nabla \times \vec{H}) = 0 \Rightarrow \nabla \cdot \vec{J} = 0$

Continuity of current → $\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$

Not consistent for time-varying conditions

Add another term

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d$$

$\vec{J}_d$: Displacement current

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Displacement Current

We have to define J_d

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d$$

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}_d$$

$$\nabla \cdot \vec{J}_d = -\nabla \cdot \vec{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

$\nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$ → Maxwell's equation for a time-varying field

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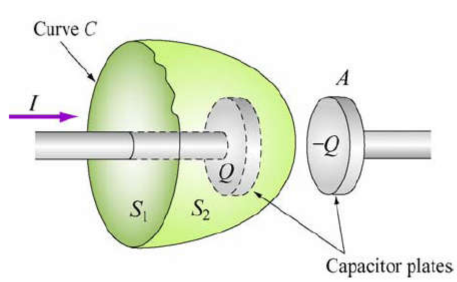
Current Through a Capacitor

A typical example of such current is the current through a capacitor when an alternating voltage is applied to its plates.

$$\oint_L \vec{H} \cdot d\vec{l} = \int_{S_1} \vec{J} \cdot d\vec{S} = I_{\text{enc}} = I$$

$$\oint_L \vec{H} \cdot d\vec{l} = \int_{S_2} \vec{J} \cdot d\vec{S} = I_{\text{enc}} = 0$$



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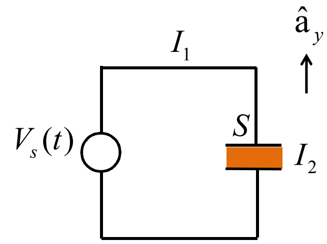
Current Through a Capacitor

Consider a capacitor connected to an alternating source voltage $V_s(t)$.

$$V_s(t) = V_0 \cos \omega t$$



Total current, $I = I_c + I_d$

a. Cross-section of the conductor:

$$I_1 = I_{c1} + I_{d1}$$

In a perfect conductor, $D = E = 0$, $I_{d1} = 0$.

$$I_1 = I_{c1} = C\frac{dV_c}{dt} = C\frac{d}{dt}(V_0 \cos \omega t) = -CV_0\omega \sin \omega t$$

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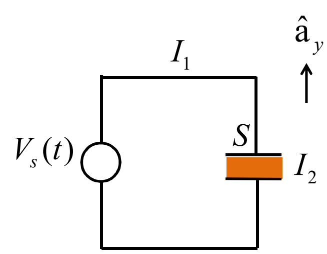
Current Through a Capacitor

b. Cross-section of the capacitor:

$$I_2 = I_{c2} + I_{d2}$$

In a perfect dielectric, $\boldsymbol{I_{c2} = 0}$.



$$\vec{E} = \hat{a}_y \frac{V_c}{d} = \hat{a}_y \frac{V_0 \cos \omega t}{d}$$

$$I_2 = I_{2d} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$= \int_S \left[\frac{\partial}{\partial t}\left(\hat{a}_y \frac{\varepsilon V_0}{d} \cos \omega t\right)\right] \cdot (\hat{a}_y dS)$$

$$= -\frac{\varepsilon S}{d} V_0 \omega \sin \omega t = -CV_0 \omega \sin \omega t$$

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Maxwell's Equations

Differential Form	Integral Form	
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot d\vec{S} = \int_v \rho_v dv$	→ Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$	→ Conservation of magnetic flux
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B} \cdot d\vec{S}$	→ Conservation of electric field
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S}$	→ Ampere's law

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Current Through a Capacitor

A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin $10^3 t$ V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_0$.

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Current Through a Capacitor

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{dV}{dt}$$

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

$$I_c = \frac{dQ}{dt} = S\frac{d\rho_s}{dt} = S\frac{dD}{dt} = \varepsilon S\frac{dE}{dt} = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

$$I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot 10^3 \times 50 \cos 10^3 t$$
$$= 147.4 \cos 10^3 t \text{ nA}$$

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