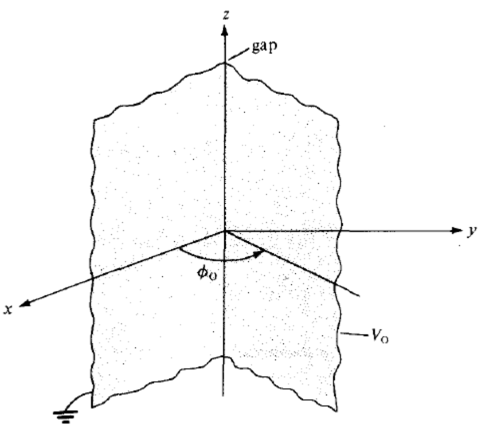
BOUNDARY VALUE PROBLEMS

1

Example – 2

Semi-infinite conducting planes $\phi = 0$ and $\phi = \pi/6$ are separated by an infinitesimal insulating gap. If $V(\phi = 0) = 0$ and $V(\phi = \pi/6) = 100$ V. Calculate V and E in the region between the planes.



2

Example – 2

In Cylindrical coordinates

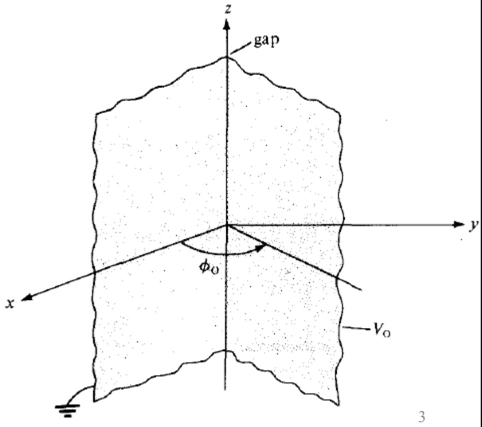
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Solve Laplace's equation

As V depends only on ϕ

$$\nabla^2 V=\frac{1}{\rho^2}\frac{d^2V}{d\phi^2}=0\Rightarrow\frac{d^2V}{d\phi^2}=0$$

$$V=A\phi+B$$



3

Example – 2

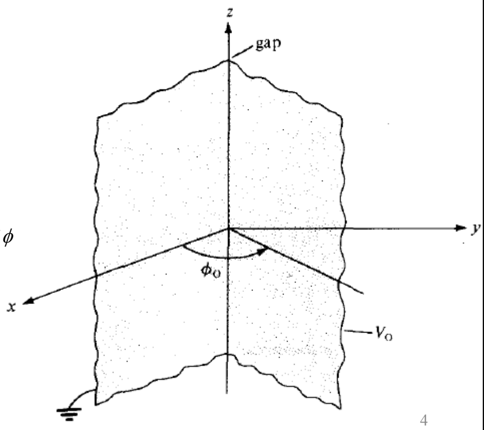
Apply boundary conditions

When $\phi=0,\ V=0\Rightarrow 0=0+B\rightarrow B=0$

When $\phi=\phi_0,\ V=V_0\Rightarrow A=\frac{V_0}{\phi_0}$

$$V=\frac{V_0}{\phi_0}\phi$$

$$\vec{E}=-\nabla V=-\frac{1}{\rho}\frac{dV}{d\phi}\hat{a}_\phi=-\frac{V_0}{\rho\phi_0}\hat{a}_\phi$$



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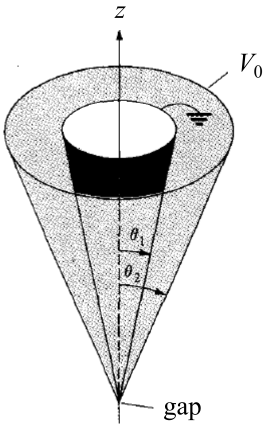
Example – 3

Two conducting cones $(\theta = \pi/10 \text{ and } \theta = \pi/6)$ of infinite extent are separated by an infinitesimal gap at $r = 0$. If $V(\theta = \pi/10) = 0$ and $V(\theta = \pi/6) = V_0$ volt, find V and $\vec{E}$ between the cones.



Example – 3

Solve Laplace's equation

V depends only on θ, $\nabla^2 V = \dfrac{1}{r^2 \sin\theta} \dfrac{d}{d\theta}\left[\sin\theta \dfrac{dV}{d\theta}\right] = 0$

$$\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0 \Rightarrow \sin\theta \frac{dV}{d\theta} = A \Rightarrow \frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

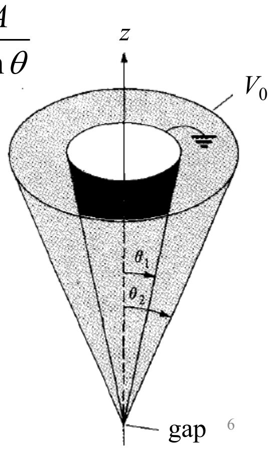
Integrating

$$V = A\int \frac{d\theta}{\sin\theta} = A\int \frac{d\theta}{2\cos(\theta/2)\sin(\theta/2)}$$

$$= A\int \frac{(1/2)\sec^2(\theta/2)d\theta}{\tan(\theta/2)}$$

$$= A\int \frac{d(\tan\theta/2)}{\tan\theta/2}$$

$$= A\ln(\tan\theta/2) + B$$



Example – 3

Apply boundary conditions

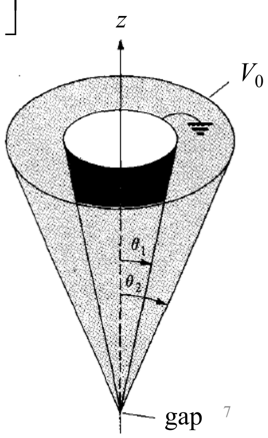
$$V(\theta=\theta_1)=0 \rightarrow 0 = A\ln(\tan\theta_1/2)+B \rightarrow B=-A\ln(\tan\theta_1/2)$$

$$V = A\ln(\tan\theta/2)-A\ln(\tan\theta_1/2) = A\ln\left[\frac{\tan\theta/2}{\tan\theta_1/2}\right]$$

When $V(\theta=\theta_2)=V_0 \rightarrow V_0 = A\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]$

$$A = V_0/\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]$$

$$V = V_0\ln\left[\frac{\tan\theta/2}{\tan\theta_1/2}\right]/\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]$$



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